IDENTIFICATION AND ESTABLISHMENT OF SOCIAL AND SOCIOMATHEMATICAL NORMS ASSOCIATED WITH MATHEMATICALLY PRODUCTIVE DISCOURSE

By

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Abstract

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For some time, the mathematics education community has sought to involve students more actively in classroom mathematical discourse, but realizing this goal has been problematic. This study has two goals: the first is to better characterize mathematically productive discourse by identifying social and sociomathematical norms that accompany it. Mathematically productive discourse is defined as discourse which holds mathematics as the authority, focuses on sense-making, and strives to create mathematical coherency. The second goal of the study is to identify strategies that teachers can use to establish these norms in their classroom.

An in-depth case study was performed of a 5th grade classroom where mathematically productive discourse regularly occurs. Twenty-five observations were performed over the course of the 2014-2015 school year, as well as twelve interviews with the teacher. Observations were video-recorded and interviews were audio-recorded. The data were analyzed using principles of grounded theory including open coding, axial coding, and the constant comparative method.

One social norm, active listening, and four sociomathematical norms, coherency, justification, computational strategies, and multiple perspectives, were identified in the classroom. To establish these norms, the teacher employed four “in-the-moment” strategies: direct prompts, normative comments,
highlighting positive examples, and modeling. However, the teacher also had a more comprehensive vision for the progression of her class’s mathematical development over the first two months of the school year. This was reflected in the way that she established a conducive classroom environment, systematically taught her students mathematical skills and practices that built upon each other, and in her concept-focused use of mathematical tasks.

The results of this study offer insight into how mathematical discourse, tasks, and practices should be conceptualized. They also underscore the importance of the teacher’s content knowledge in enabling these norms to emerge.
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Chapter 1: Rationale

How do people learn? Arguably this question, investigated in the well-known National Research Council book *How People Learn*, is the most fundamental question in education. Fostering learning is a foundational goal of education and is often the standard by which its efficacy is assessed. If students demonstrate evidence of having learned the desired skills and content knowledge, then the educational system is usually deemed successful. Otherwise it is deemed inadequate, often precipitating a call for reform. Because of the central importance of learning, other aspects of education, such as pedagogy, curriculum, task selection, class size, and use of technology, are commonly evaluated by their effect on learning. Changes in these areas tend to be justified by a purported beneficial impact on learning.

A theory of how people learn then carries far reaching implications for every area of education. For most of the 20th century, education was dominated by the behaviorist learning theory (Sperry, 1993). Behaviorism attempted to make psychology an objective, hard science by focusing strictly on observable evidence, i.e. behavioral evidence (Miller, 2003). Talk of mental events or subjective mental states was thereby precluded due to their lack of observability. During the 1970s, behaviorism’s supremacy in the field of psychology waned rapidly as a new paradigm called cognitivism arose (Sperry, 1993). In the following decade, cognitivism’s influence spread beyond the field of psychology into many other fields, such as education (Sperry, 1993). Cognitivism abandoned behaviorism’s restrictions on non-observable mental phenomena and legitimized the notions of consciousness and theories about inner mental processes. The behaviorist view of learning as a change in behavior was rejected in favor of a new perspective that emphasized “what learners know and how they acquire it” (Jonassen, 1991, p. 6).

The widespread acceptance of cognitivism in the 1970s allowed a new theory of learning called constructivism to rapidly gain influence (Steffe & Kieren, 1994). Although constructivism influenced many different fields, I focus here solely on its impact in the field of math education within the United States. Constructivism’s rise to prominence began with the publishing of several key papers by Ernst von
Glasersfeld in 1980 and 1981. The theory’s influence grew so quickly that some have deemed this period in the 1980s as the *constructivist revolution* (Steffe & Kieren, 1994). By the mid-1990s, constructivism was acknowledged, even by its critics, to be a major, if not the dominant, theory of learning in the mathematics education community (Lerman, 1996). But what exactly is constructivism? The answer is surprisingly complicated because, as Phillips (1995) notes, constructivism comes in many different varieties. Nevertheless, all such varieties are founded on the epistemic claim that knowledge does not exist external to individuals (Phillips, 1995). Constructivism’s foundational assumption is that an individual constructs all of his or her own knowledge, rather than receiving it from external sources. As a sense-making being, the individual is constantly creating mental models, called *schemas*, in an attempt to organize experiences into a coherent whole. Experiences are either fit into an existing schema in a process called *assimilation*, or the schema is modified to fit an unexpected experience (a *perturbation*) in a process called *accommodation* (von Glasersfeld, 1989a). Learning is defined as the accommodation that occurs when a perturbation is encountered (von Glasersfeld, 1989a). An important implication of constructivism is that the individual necessarily plays an active role in learning as he or she continually builds and modifies schemas. Most constructivists agree that social interactions play a primary role in providing perturbations that prompt the individual to construct new knowledge (Perkins, 1999).

**Forms of constructivism**

One of the earliest manifestations of constructivism to gain prominence in math education was radical constructivism, promoted heavily by Ernst von Glasersfeld (1984, 1989a, 1989b, 1995). Radical constructivism made the additional claim that there is no objective reality external to and independent of the individual (von Glasersfeld, 1995). Truth as a correspondence with an external reality was replaced by the pragmatic concept of viability: “truth” was now what works for an individual’s subjective reality. Other constructivists rejected von Glasersfeld’s skepticism about an independent reality external to any particular individual. These constructivists, whom von Glasersfeld (1989b) deemed “trivial
constructivists” (p. 114), held that individuals do exist within a reality external to themselves. Schemas are modified then not merely to better match experiences but to better match reality.

Still other constructivists, dissatisfied with the individualistic focus of constructivism, combined constructivism with sociocultural theory to create a pragmatic blend of the two, called the emergent perspective (Cobb & Yackel, 1996). Whereas constructivism focuses on the individual mind and how it creates knowledge, sociocultural theory places primary focus on the broader sociocultural community (Cobb, 1994). From this perspective, learning occurs as individuals are enculturated into the practices of a larger community. Indeed, sociocultural theory claims that higher mental functions themselves are the result of individuals internalizing their social activities (Cobb, 1994). This means that the individual mind has its origins in the community as social interactions shape not only the content of individual thought, but the very nature of individual thinking itself. The emergent perspective holds both constructivism and socioculturalism, acknowledging that both the individual mind and the social community are key sources of knowledge, with neither one more ultimate than the other (Cobb & Yackel, 1996). One theory may temporarily receive a greater emphasis than the other depending on where the researcher’s focus lies. If the researcher focuses on the individual within the group, the constructivist theory receives greater emphasis, but if the researcher focuses on the group as a whole, then the sociocultural theory receives the emphasis. The emergent perspective holds that the two theories are in reality two different perspectives depending on whether one chooses to consider the group primarily as a single unit or as a collection of individuals. This means that the emergent perspective holds the two theories of constructivism and socioculturalism in a pragmatic way (Cobb & Yackel, 1996). Taken literally, the two theories are contradictory: one claims that the individual is ultimate and shapes the group (constructivism), while the other claims that the group is ultimate and shapes the individual (socioculturalism). Clearly, both of these claims cannot simultaneously be true. However, when they are seen in a new light, as merely perspectives or metaphors for learning rather than absolute truth, there is no contradiction.
Implications and effects of constructivism

Since constructivism’s rise to prominence, many teachers, researchers, and others in the field of math education have attempted to determine its implications for pedagogy, a task that has proven more difficult than might be anticipated. Simon (1995) observed that while constructivism provides a useful framework for thinking about learning, “it does not tell us how to teach mathematics; that is, it does not stipulate a particular model” (p. 114). In essence, Simon highlighted the fact that constructivism is a theory of learning and not of teaching. von Glasersfeld (1995) agreed with this when he wrote that constructivism “[can] not produce a fixed teaching procedure. At best it may provide the negative half of a strategy” (p. 177). However, many constructivists, including Simon and von Glasersfeld, agreed that constructivism does carry general implications for pedagogy. Since individual students actively construct knowledge, the teacher should continually strive to understand what schemas the students are constructing in response to classroom activity. Student mistakes are an opportunity for exploration rather than immediate evaluation since they offer insight into how students are attempting to make sense of their experiences (von Glasersfeld, 1989a). More classroom discourse in general also aids the teacher in discovering student schemas. Since knowledge cannot be directly transmitted via language, the teacher should place less emphasis on direct instruction (which puts the students in a passive role) and greater emphasis on active student engagement in classroom tasks (von Glasersfeld, 1995). Social interactions are the most common source of perturbations (which then result in learning), so teachers should allow students to collaborate more frequently in discussing and solving problems (von Glasersfeld, 1989a). Students are naturally sense-making beings, so repeated drill of procedures and algorithms, devoid of understanding, should be avoided. Or at the very least, teachers should recognize a distinction between performance in executing a procedure and conceptual understanding. Finally, teachers should recognize there may often be many different valid ways of solving a problem (von Glasersfeld, 1989a).

Despite these general implications however, constructivism still leaves quite a bit unsaid regarding specifics of the math teacher’s role in the classroom. As teachers have tried to adapt their pedagogy to be consistent with constructivism, one particular issue has arisen repeatedly over the years,
usually playing itself out in the following manner: The teacher has particular content knowledge that he or she would like students to attain. In keeping with constructivist principles, the teacher wishes to avoid direct instruction and instead allow the students a more active role in the classroom, with the goal that the students will eventually construct the desired content knowledge for themselves. However, allowing the students a more active role in the classroom inevitably brings greater unpredictability (Simon, 1995). Perhaps the students hit a conceptual roadblock and give up. Perhaps they focus on an irrelevant detail that leads them on an unrelated tangent. Or perhaps they reach a contradictory conclusion without being aware of it. The teacher then struggles with how to bring the students to construct the desired content knowledge without simply reverting to direct instruction and thereby placing the students in a passive role. A number of researchers, including Ball (1993), Chazan and Ball (1999), Elmore, Peterson, and McCarthey (1996), Sherin (2002), Simon (1995), and Williams and Baxter (1996) and have observed this issue occurring in some form. The dilemma became so prominent that Elmore, Peterson, and McCarthey (1996) labeled it the constructivist dilemma.

Due to the creation of the 1989 Curriculum and Evaluation Standards for School Mathematics (Standards) by the National Council of Teachers of Mathematics (NCTM), the constructivist dilemma was brought to the forefront in the field of mathematics education in a particularly discursive form. The 1989 Standards was a highly influential document that outlined a new vision for mathematics classrooms. In 1991, NCTM released the follow-up Professional Standards for Teaching Mathematics (Professional Standards) to further elaborate on the kind of teaching that would support the vision in the Standards. Although the term “constructivism” was never explicitly used, both documents contained strong articulations of constructivist assumptions.

In many classrooms, learning is conceived of as a process in which students passively absorb information, storing it in easily retrievable fragments as a result of repeated practice and reinforcement. Research findings from psychology indicate that learning does not occur by passive absorption alone (Resnick 1987). Instead, in many situations
individuals approach a new task with prior knowledge, assimilate new information, and construct their own meanings. (NCTM, 1989, p. 10)

Also, the idea that perturbations elicit construction of knowledge was reflected when the Standards stated that “[children] will accept new ideas only when their old ideas do not work or are inefficient” (p. 10). The individual’s innate striving for coherence in their schemas, another constructivist assumption, was also acknowledged: “ideas are not isolated in memory but are organized and associated with the natural language that one uses and the situations one has encountered in the past” (NCTM, 1989, p. 10). Finally, the constructivist intentions of the NCTM’s vision were unequivocally acknowledged: “This constructive, active view of the learning process must be reflected in the way much of mathematics is taught” (NCTM, 1989, p. 10). Throughout the entire document, the Standards repeatedly stated that students construct their own knowledge. For example, “young children are active individuals who construct, modify, and integrate ideas by interacting with the physical world, material, and other children. Given these facts, it is clear that the learning of mathematics must be an active process” (NCTM, 1989, p. 17). Mathematics itself was referred to as a “human creation” (NCTM, 1989, p. 128). Given this clear evidence, the constructivist foundation of the Standards and Professional Standards is undeniable.

**Productive discourse**

One of the more prominent emphases in the Standards was that student learning occurs through mathematical discourse. Some researchers even called discourse a “central tenet” (Williams & Baxter, 1996, p. 22) of the document. Communicating mathematically was one of five main overarching goals for students listed in the introduction of the Standards. “The very act of communicating clarifies thinking and forces students to engage in doing mathematics. As such, communication is essential to learning and knowing mathematics” (NCTM, 1989, p. 214). In keeping with this communication goal, the Standards offered more detailed discursive goals for various grade levels. In the years since the Standards and Professional Standards were first published, many researchers have attributed an increased interest in discourse to these reform documents. Silver and Smith (1996) observed that because of the Standards and
Professional Standards, “interest in communication is both more widespread and more central to mathematics education reform efforts than ever before” (p. 20). Hufferd-Ackles, Fuson, and Sherin (2004) made the following observation:

The successful implementation of [NCTM] reform requires that teachers change traditional teaching practices significantly and develop a discourse community in their classroom…. research [over the past decade] describes the many dilemmas that teachers face in trying to implement [the NCTM vision], and more specifically in establishing a discourse community. (pp. 81-82)

More recently, Baxter and Williams (2010) wrote that “over the past two decades… suggested [NCTM] reforms in curriculum, instructional methods, and assessment techniques have become the focus of both research and policy discussions in the United States” (p. 7), and then immediately proceeded to mention the emphasis on discourse. It is reasonable to conclude then that the NCTM’s vision has prompted a continuing emphasis on mathematical discourse within the math education community.

However, the NCTM never gave their envisioned discourse a specific designation. Instead, they described it by listing its qualities. For example, the source of authority for this discourse was to be mathematical reasoning rather than teacher or textbook. “That something is true because the teacher or the book says so is the basis for much traditional classroom discourse. Another view, the one put forth here, centers on mathematical reasoning and evidence as the basis for discourse” (NCTM, 1991, p. 34).

Another quality of the envisioned discourse was its emphasis on sense-making rather than evaluation of student ideas. “If the discourse is to focus on making sense of mathematics, on learning to reason mathematically, teachers must refrain from calling only on students who seem to have right answers” (NCTM, 1991, p. 36). Yet another quality of the discourse was its striving for mathematical coherency.

Although it is often necessary to teach specific concepts and procedures, mathematics must be approached as a whole. Concepts, procedures, and intellectual processes are interrelated. In a significant sense, "the whole is greater than the sum of its parts." Thus, the curriculum should include deliberate attempts, through specific instructional
activities, to connect ideas and procedures both among different mathematical topics and with other content areas. (NCTM, 1989, p. 11)

I have defined the term *mathematically productive discourse* (or more simply, *productive discourse*) to be discourse characterized by these three qualities. Firstly, productive discourse holds mathematical reasoning as its authority. This means students appeal to mathematical reasoning as a basis, rather than the teacher, textbook, social status, or other nonmathematical reasons. Secondly, productive discourse focuses on sense-making. This means it helps students to personally understand the *meaning* of the mathematics that are doing. Finally, productive discourse strives to create coherency by explaining various mathematical topics, problems, or solution strategies in terms of others.\(^1\) It is worth noting that the particular *form* that this discourse might take is left unspecified. The *Professional Standards* noted that such discourse can take many forms including “making conjectures, proposing approaches and solutions to problems, and arguing about the validity of particular claims” (p. 45), as well as “verify[ing], revis[ing], and discard[ing] claims” (p. 45). When studying mathematical discourse, many researchers have focused on similar discursive qualities. For example, Baxter and Williams (2010) specifically looked for “the extent to which discourse focused on sense-making” (p. 14) as well as “the site of intellectual authority” (p. 14). Hufferd-Ackles, Fuson, and Sherin’s (2004) conception of desirable mathematical discourse was one where “math sense becomes the criterion for evaluation” (p. 88). Connor, Singletary, Smith, Wagner, and Francisco (2014) emphasized mathematical coherency when they stated that “to facilitate productive mathematical discussions, teachers must engage in behavior that helps students build from their own understandings toward appropriate understandings of mathematical ideas” (p. 402).

Despite the focus on communication within the math education community, teachers have not always been successful in their efforts to create productive discourse in their classrooms. In describing the teacher’s discursive role, the NCTM reform documents tended to use general terms such as guide, moderator, facilitator, and questioner. They also employed many negative exhortations, cautioning

\(^1\) The rationale for choosing these three qualities as definitional will be explained in chapter 2 when the literature is reviewed.
teachers against using direct instruction or dispensing information to their students. Ball (1996) has pointed out that such general descriptions are too vague to be of real help to teachers:

[The NCTM documents] do not provide guidance on the specifics of day-to-day, minute-to-minute practice…. When is a disagreement among students worth continuing? When should a teacher step in and clear up a controversy? When is a particular student’s statement best left alone? When is it good to probe? (para. 17)

Chazan and Ball (1999) have also noted that the repeated negative exhortations not to use direct instruction nor dispense information are inadequate because they ignore the significance of context and seem to imply a passive role for the teacher rather than creating a positive vision for the teacher to work towards. A number of studies have documented classrooms where teachers successfully increased the amount of student discourse while failing to foster the qualities of productive discourse. Williams and Baxter (1996) observed a middle-school classroom where, because of the teacher’s efforts, students actively participated in discourse. However, they also noted that this discourse suffered from many serious issues.

It became obvious that discourse qua discourse did not lead to the production of common knowledge in the way that the teacher intended. In some cases, the discourse became for students an end in itself. In others, it became another superfluous requirement. In each example, the social scaffolding… became for the students part of the meaningless ritual of classroom life rather than a tool for learning. In this way, it did not support… the creation of useful mathematical knowledge. (p. 36)

Nathan and Knuth (2003) observed a middle school teacher who specifically attempted to increase the frequency of student participation in discourse while simultaneously reducing her own. She was successful in accomplishing her goal, but the resulting discourse often had “a lack of rigorous argumentation and evidence…. Student ideas were offered publicly for others to pick up, refute, or ignore, often with no basis for evaluation other than opinion” (Nathan & Knuth, 2003, pp. 199-200). Chazan and Ball (1999), despite their best efforts to promote student discourse in a high school algebra class, noted
that the students “tended to become frustrated… and would either turn to [the teacher] to tell them who was right and who wrong [sic] or would try to intimidate everyone into agreeing with them” (p. 3). These examples all serve to illustrate the complexity and difficulty of creating productive mathematical discourse.

**Social norms**

In analyzing discourse, many researchers have used the construct of *social norm*. As Yackel and Cobb (1996) explain, social norms are the group correlate of the individual’s beliefs about people’s roles. Social norms are a construct both descriptive and evaluative in nature: they describe regularities in group social behavior and function as rules or criteria by which to judge acceptable social behavior. How do these social norms form? They arise organically out of the group’s social behavior and social expectations. As Sfard (2000) points out, group behavioral regularities and group social expectations evolve concurrently. From one vantage point, social expectations arise out of behavioral regularities. As Sfard (2000) explains, “by incessantly repeating themselves, the unwritten and mostly unintended [social] rules shape people’s conceptions of “normal conduct” and, as such, have a normative impact” (p. 170). Simply put, as we repeat the same behaviors again and again, we come to expect them. From a different vantage point however, behavioral regularities arise out of commonly accepted social expectations. Without commonly accepted social expectations, group behavioral regularities would never occur. Sfard (2000) argues that effective group interaction would not even be possible without some degree of alignment in social expectations. Social expectations “have an enabling effect in that they eliminate the infinite possibilities of discursive moves and leave the interlocutors with only a small number of reasonable choices. Without this preselection, we might be deprived of the ability to participate in any discourse” (p. 172). Hence, group behavioral regularities and group social expectations evolve in tandem. They can be conceptualized as different perspectives of the same phenomenon. This phenomenon is encapsulated by the construct of social norm. Note that this definition of social norm does not include an affective component. Two groups that display the same behavioral regularities and expectations might be
experiencing very different inward states. One group might be carrying out these behavioral regularities joyfully and whole-heartedly, while another might be exhibiting the same regularities with inward resentment towards them. Whether regularities are expressed sincerely or grudgingly, and whether expectations are held enthusiastically or reluctantly, is irrelevant for purposes of norms. If two groups demonstrate the same behavioral regularities and expectations, then they will be said to hold the same norms.

But what does it really mean to talk about “group behavioral regularities”? A group is composed of individuals. When we use the term “group”, we are taking multiple individuals and viewing them as a single collection. Likewise, a group’s characteristics are comprised of many individuals’ characteristics that are being viewed as a unity. But how accurate is this kind of language? For example, if we say a group of 20 people “demonstrate” Behavior X, are we saying that every one of the twenty individuals demonstrates Behavior X? If only 19 demonstrate it and 1 did not, is it still accurate to say that “Behavior X is a group regularity”? Certainly, if only 5 people demonstrate Behavior X and 15 never do, it is not accurate to say that “Behavior X is a group regularity”. At what point then do we say “Behavior X is a group regularity”? What I am highlighting is that there is a certain amount of unavoidable ambiguity when discussing a group’s characteristics. Generalizing about a group of people is never an exact science. I mention this to alert the reader to the issue, not because I have developed a flawless solution. I do not intend to create an arbitrary cut-off point to determine group characteristics (such as “if 60% of the group members demonstrates this behavior, then it will be classified as a group regularity,” or “if Behavior X is demonstrated at least once per day by someone in the group, then I conclude that it is a group regularity”). Instead, I will rely on my subjective judgment in determining the regularities that seem to be demonstrated by “almost everybody” (Sfard, 2007, p. 539) in the group.

Does everyone in the classroom contribute equally to social norm formation? As the person wielding the most influence in classroom discussions, the teacher “makes the decisive contribution” (Sfard, 2000, p. 172) to shaping classroom social norms. This is because the teacher has the authority within the group to evaluate all student social contributions. By affirming or reprimanding certain student
behaviors, the teacher more than any other individual shapes the students’ social expectations. This is not to say that students do not also influence the group’s social norms. They do, but to a lesser extent than the teacher since they are under the teacher’s evaluative authority. For example, Bart may believe that it is not necessary to raise his hand before speaking, so he abruptly speaks out in class. The teacher then reminds Bart to raise his hand. Ultimately, Bart did contribute to reinforcing a social norm by providing an example of unacceptable behavior. Bart might not have originally intended to make this sort of a contribution, but because of the teacher’s evaluative authority, Bart ended up doing so. Alternatively, Bart could have raised his hand and the teacher then might have praised him for doing so. Both of these examples demonstrate how students and teachers contribute to classroom social expectations. However, the teacher contributes more influentially than the students because he or she has the power to shape the nature of the students’ contributions by interpreting and evaluating their actions. What if the teacher does not evaluate a student’s action? Assume that Bart speaks out without raising his hand and the teacher fails to comment on it. In this case, by remaining silent on the issue, the teacher has tacitly approved of Bart’s conduct and Bart has helped to establish this behavior as acceptable.

Once social norms have been established, do they continue to change or are they static? Group regularities and social expectations may certainly evolve over time, so social norms are never “set” in an immutable way. Practically however, researchers have observed that most classroom social norms are established within the first two months of the school year and maintain a sort of equilibrium for the remainder of the school year as the class settles into well-established social patterns (McClain & Cobb, 2001; Wood, 1999; Wood, Cobb, & Yackel, 1991).

Are social norms directly observable? Recall that a social norm is a construct that is both descriptive and evaluative in nature: it describes regularities in group social behavior and also functions as a standard for acceptable group behavior. Earlier, I explained how these two aspects of a social norm evolve concurrently: regular behavior begets social expectations, and social expectations beget regular behavior. We could also say that the behavioral regularities are the standard for acceptable group behavior. Group social behavior is directly observable, although classifying “regular” group behavior
requires some degree of interpretation on the observer’s part. Are social expectations observable? If certain circumstances arise, then yes. If a particular group member violates a social norm, the reaction of the other members will indicate that a violation has occurred, thus revealing their expectations. For example, if students in a classroom regularly raise their hands and wait to be called on by the teacher before speaking, this behavioral regularity reflects a social norm. Assume then that Homer begins talking without raising his hand and other student immediately correct him and remind him to raise his hand. Their reactions indicate a social expectation has been violated. In this case then, both the descriptive and evaluative components of the norm are visible. However, if the social norm is never violated, the evaluative nature of it may never be directly observed.

Are group members consciously aware of social norms? In considering this question, it will be helpful to consider an individual first. Recall that social norms are group correlate of the individual’s beliefs. Some individuals tend more towards introspection and may spend a great deal of time deliberately reflecting on their own beliefs. If we asked such an individual what they believe, they would likely be able to give us a detailed explanation of their beliefs. Other individuals however are less introspective and may have never deliberately reflected on what they believe is appropriate social behavior. If we asked such a person what they believe about social behavior, they may provide a vague answer, or even say that they are unsure. Does this mean that this person does not have any beliefs about others’ roles? No. Imagine that we interview two people, Thoughtful Theodore and Inarticulate Ian, about their beliefs regarding use of obscenities. Assume both individuals never use obscene language and object to others’ use of it. Assume also that Thoughtful Theodore has often reflected on what he believes and when asked, provides a well-articulated explanation of his belief that obscene language should not be used. Finally, assume that Inarticulate Ian is not one for reflection and has never taken time to deliberately consider what he believes regarding social behavior. When other people use obscenities, Ian feels an instinct to verbally object, but has never consciously connected this instinct with a belief. When asked about his beliefs concerning use of obscenities, Ian simply shrugs and says “he doesn’t like them.” What should we conclude about Theodore’s and Ian’s beliefs regarding obscenities? I argue that both hold the same belief
in this case. However, Thoughtful Theodore is more self-aware of his beliefs. My point in using this example is to illustrate that people vary in the extent to which they are self-aware of their social expectations. Some, through extensive reflection, are acutely aware of their expectations. The less self-aware may experience their social expectations as an impulse or an instinct, something that “feels right” or “feels appropriate”, but remain unable to accurately articulate this expectation. Regardless of their level of self-awareness, if both individuals hold the same social beliefs, an observer should witness similar regularities in their social behavior. So to clarify, when I say that an individual is not self-aware of their social beliefs, I am not saying that the individual does not have social beliefs, nor am I saying that the individual is not influenced by those social beliefs. Rather, I am saying that the individual regularly acts upon those beliefs in an automatic fashion without consciously recognizing them.

The question may be raised at this point: what if the individual articulates beliefs that are inconsistent with their actions? For example, say that Inarticulate Ian claims that obscene language should not be used by anyone and then proceeds to regularly use obscene language. What should we conclude in such a situation? I posit that an individual’s actions reflect their true beliefs and overrule professed beliefs should a contradiction arise. So I would conclude then that Ian believes obscene language is acceptable and exhibits no self-awareness of that belief.

Having considered the extent to which individuals may be self-aware of their beliefs, I now consider group awareness of social norms. I argue that the observations made about individuals apply to groups as well. Every group has social norms, but groups may vary in their self-awareness of these norms. Some groups may have explicitly reflected upon and discussed their social norms, while other groups may have never done this. Even within a group, some members may be more self-aware of norms than others.

Research indicates that certain social norms seem to be necessary for, or at the very least correlated with, the mathematically productive discourse mentioned earlier. For clarification, I refer to such social norms as productive social norms. The expectation that students explain and justify their

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2 In my personal experience, I have found that it often takes novel social situations in order to expose expectations of which I was previously unaware.
thinking is a productive social norm (Tatsis & Koleza, 2008; Kazemi & Stipek, 2001; McClain & Cobb, 2001; Yackel, Rasmussen, & King, 2000), as is student collaboration in tackling problems (Tatsis & Koleza, 2008; Kazemi & Stipek, 2001; Yackel, Cobb, & Wood, 1991). An essential part of this collaboration is the norm of active listening, in which students try to make sense of others’ ideas and challenge those ideas if they disagree (McClain & Cobb, 2001; Yackel, Rasmussen, & King, 2000; Yackel & Cobb, 1996). Other productive social norms include comparing multiple solution paths to problems (Kazemi & Stipek, 2001; Yackel, Rasmussen, & King, 2000), viewing mistakes as a normal and acceptable part of the learning process (Kazemi & Stipek, 2001; McClain & Cobb, 2001), persistence in solving challenging problems (Yackel, Cobb, & Wood, 1991), and differentiating between a mathematical disagreement and a personal disagreement (Wood, 1999).

Most students do not bring an understanding of these productive social norms to the classroom because they differ from the norms of everyday discourse. Lampert, Rittenhouse, and Crumbaugh (1996) provide examples of students who confuse mathematical argumentation with social negotiation. Some of these students tried to argue for the validity of their answers based on social status, penmanship, or rhetoric, while others acted as if the most important goal was to maintain congeniality and leave every mathematical idea unchallenged and untested. Lampert, Rittenhouse, and Crumbaugh (1996) conclude that “considering all assertions to be tentative and open to reasoned challenge from one’s peers… seems to go hand in hand with damaged egos and feelings that one is being treated in a ‘mean’ way” (p. 757). Their study underscores the need for teachers to intentionally develop productive social norms as students do not typically learn these norms from everyday discourse.

The presence of productive social norms in a classroom does not guarantee that productive discourse will occur. Multiple studies documented classrooms where productive social norms were present and student discourse was plentiful yet mathematically unproductive. Nathan and Knuth (2003) described a teacher who, in order to allow students opportunity to regularly share their thinking, maintained a relatively non-impositional discursive policy. The result was a chaotic discursive atmosphere where “student ideas were offered publicly for others to pick up, refute, or ignore, often with
no basis for evaluation other than opinion” (pp. 198-199). McClain and Cobb (2001) observed a teacher who wanted students to feel comfortable sharing multiple solution strategies, but ended up with many student suggestions that were repetitive or failed to make a substantive contribution to the discussion at hand. “Whole-class discussions appeared to us to involve a series of disjoint turns… that did not contribute to the mathematical agenda” (p. 247). Kazemi and Stipek (2001) found two 4th grade classrooms that, despite having productive social norms, had a relatively low press for conceptual thinking. Likewise, Williams and Baxter (1996) documented a similar classroom in which student discourse was mathematically superficial and ritualized, despite the presence of productive social norms. Together, these studies demonstrate that productive social norms alone do not suffice to produce mathematically productive discourse.

**Sociomathematical norms**

The construct of *sociomathematical norm* was first promulgated by Yackel and Cobb\(^3\) (1996) and resulted from a consideration of the relationship between social norms and mathematical argumentation. Yackel and Cobb (1996) define sociomathematical norms as “normative aspects of mathematics discussions specific to students’ mathematical activity” (p. 461), indicating that sociomathematical norms are a special subset of social norms that apply uniquely to mathematics. Yet this definition seems somewhat vague and unsatisfying. “Students explain their thinking when providing an answer” would be a social norm according this definition, since there is nothing in it specific to mathematics. But what about “students explain their *mathematical* thinking when providing an answer”? This is essentially the same social norm as before except that it has now been limited to a mathematics context. Would it now be a sociomathematical norm? Is it now “specific to students’ mathematical activity?” (Yackel & Cobb, 1996, p. 461).

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\(^3\) I use the phrase “first promulgated by” because technically, Voigt (1995) was the first to coin the term *sociomathematical norm*. However, Yackel and Cobb’s (1996) work is the most well-known and most commonly cited work regarding sociomathematical norms, leading many to assume that they invented the term.
In an attempt to further clarify Yackel and Cobb’s (1996) definition, I propose the following definition: sociomathematical norms are a subset of social norms that necessarily require specific mathematical content knowledge in order to be understood. For example, “students should explain their mathematical reasoning” would be a social norm. It details how students should interact with others and talk about mathematics. However, students do not necessarily require any specific mathematical content knowledge to understand what it means to explain their reasoning. However, a more specific expectation of what constitutes an acceptable explanation, such as “explaining your reasoning means appealing to a mathematical basis of previously established facts, theorems, and relationships” necessarily requires certain mathematical content knowledge. To understand this sociomathematical norm, students must understand what a mathematical basis is and how to appeal to it. Thus, the statement “students should explain their reasoning by appealing to a mathematical basis consisting of established facts, theorems, and relationships” is a sociomathematical norm.

To further illuminate the difference between social and sociomathematical norms, I now provide several more examples. “Students should find multiple different solutions” would be a social norm because no specific mathematical content knowledge is required to understand what this norm requires. A more detailed expectation of what constitutes a different solution might transform this social norm into a sociomathematical norm. The statement “students should find multiple different solutions by decomposing and recomposing the numbers differently” would be a sociomathematical norm. To understand what this norm requires, students must understand what it means to decompose and recompose numbers. The statement “mistakes should be used to explore reasoning” is a social norm. However, a clarification of what kind of reasoning should be explored, such as proportional reasoning, would make this a sociomathematical norm as students must understand what proportional reasoning is in order to understand what the norm requires. Thus, “mistakes should be used to explore proportional reasoning” would be a sociomathematical norm.

These examples illustrate a key relationship between social and sociomathematical norms. Sociomathematical norms are the more specific of the two and usually consist of a social norm with an
additional clarification that invokes mathematical content knowledge. In the examples given above, the sociomathematical norms clarified more specifically what explaining reasoning means, what a different solution entails, and what exploring incorrect reasoning looks like. In each of these cases, different pieces of mathematical content (basis, decomposing/recomposing, proportional reasoning) were invoked as the criteria by which to make this clarification. These examples then illustrate how sociomathematical norms complement and clarify social norms.

While I believe that my definition helps to clarify Yackel and Cobb’s (1996) distinction between social and sociomathematical norms, there is still a certain amount of unavoidable ambiguity between the two. Consider the following statements:

a) Students should show their work.

b) Showing work means creating a visual representation.

c) Showing work means creating a visual representation that illustrates the mathematical generality of the claim.

By my definition, (a) would clearly be a social norm, but what about (b)? It would appear to be a social norm as well, since no specific content knowledge is required to understand what a visual representation is. But what if this visual representation is specified to be a graph on the coordinate plane? Then specific content knowledge would be required and hence it would become a sociomathematical norm. Now consider (c). Understanding mathematical generality, such as the domain over which a claim applies, is a key (and often difficult) concept in mathematics. Hence, (c) would be a sociomathematical norm. But what if the word mathematical was removed from (c)? Generalizing is not an activity specific to mathematics. We make generalizations about people, activities, and other things in everyday life. Therefore, it seems that (c) would not be a sociomathematical norm if the word mathematical were removed.

A similar issue arises with the statements below.

d) Students should explain their thinking.

e) Explaining thinking means providing computations.
f) Explaining thinking means providing computations within the context of the original problem. The first statement is a social norm, but the remaining two are more difficult to determine. Do students need specific mathematical content knowledge to understand (e)? They certainly must understand what computations are. Although this is an extremely basic piece of content knowledge, it is nonetheless mathematical content knowledge, so I would consider (e) to be a sociomathematical norm. However, others might consider the concept of computation to be so basic that they might classify (e) as a social norm. Statement (f) is similar to (e) but it includes the ability to contextualize one’s computations. Is this ability to contextualize computations part of mathematical content knowledge? I would argue yes for the same reason given for (e).

These examples all serve to illustrate that the boundary between social and sociomathematical norm can be very thin and even debatable at times. A minor clarification or specification can often be the difference between the two. In other cases, it is questionable whether a certain skill or piece of knowledge, such as understanding generality, computations, and contextualization, is specific to mathematics or not. In categorizing norms, I will attempt to clarify if I think a particular categorization is debatable and why so that the reader may make an informed decision.

Research suggests that sociomathematical norms may determine whether productive social norms actually lead to mathematically productive discourse. In the classroom that McClain and Cobb (2001) observed, as mentioned earlier, the teacher repeatedly asked for different solution strategies, but students would often contribute repetitive ideas, resulting in discourse “that did not contribute to the mathematical agenda” (p. 247). The teacher stressed that she wanted different solutions, but failed to clarify her criteria for what different entailed. This teacher had established a clear social norm but had failed to clarify a corresponding sociomathematical norm, limiting the effectiveness of the social norm. Eventually, the teacher explicitly clarified what constituted a different solution. This led to the classroom discourse becoming more productive as students now had mathematical criteria to appeal to as an authority in justifying whether or not their solutions were different from others’. Kazemi and Stipek (2001) investigated four elementary classrooms that all demonstrated productive social norms and found that
differing sociomathematical norms explained whether a classroom had a high or low emphasis on conceptual thinking.

Similarly to how I defined productive social norms earlier, I now define productive sociomathematical norms as sociomathematical norms that seem to be correlated with productive mathematical discourse. Known productive sociomathematical norms include the expectations that students explain the rationale for their computations rather than just summarizing the computations themselves (Clement, 1997), compare their solution strategy to others’ using explicitly defined criteria (McClain & Cobb, 2001), use mistakes to explore mathematical thinking and contradictions (Kazemi & Stipek, 2001), and come to a group consensus through mathematical argumentation during collaboration (Kazemi & Stipek, 2001).

The process of establishing norms

Many of the beliefs and expectations that students tend to bring into the classroom are not conducive to productive discourse. This was already discussed in part earlier when it was noted that students naturally tend to conflate mathematical criticism with personal criticism (Lampert, Rittenhouse, & Crumbaugh, 1996). However, the process of trying to change unproductive norms can require substantial effort on the teacher’s part. Hufferd-Ackles, Fuson, and Sherin (2004) observed a third-grade classroom where students began the school year accustomed to providing unjustified answers. Shortly into the school year, the teacher began making a deliberate effort to prompt students to share their process of reasoning along with their answer. These initial attempts were “laborious” (p. 98) and required great patience from the teacher as she often had to repeatedly prompt the students for each part of their reasoning. Other research indicates that students may actually resist the teacher if he or she attempts to change the classroom norms too abruptly. Smith (2000) provided an example of a ninth-grade teacher whose students were accustomed to math problems that required following a known procedure. When the teacher gave the students a non-routine problem, they quickly grew frustrated, gave up, and began pressing the teacher to provide more guidance. It is likely that these students had established expectations
about what it meant to “do math”, namely, following procedures. In presenting a non-routine problem, the teacher was introducing a situation that conflicted with these expectations. Unsurprisingly, the students resisted. The presence of competing norms often leads to a variety of communicative conflicts (Sanchez & Garcia, 2014). Students may assume different criteria when justifying a solution strategy or use the same word with different understandings of its meaning (Yackel & Cobb, 1996). Or, as in Smith’s (2000) example, students may not recognize the importance or legitimacy of an activity. These examples illustrate that the process of establishing productive norms can be difficult. It may require great effort on the teacher’s part and even face student resistance. In light of this fact, it is important that the teacher adopts effective strategies when trying to establish productive norms.

**The research questions**

Near the end of their article, Kazemi and Stipek (2001) made the following remark:

Although we propose that at least four sociomathematical norms worked together to create a press for conceptual learning, continued research may reveal other norms that contribute to a high press. It is also important to investigate, with longitudinal data, how sociomathematical norms are created and sustained, and how they influence students’ mathematical understanding. (p. 79)

Since this remark was made, many studies have focused on social and sociomathematical norms, as well as their formation. However, the question remains: are there other social and sociomathematical norms, yet to be identified, that are associated with the qualities of mathematically productive discourse? Most of the studies investigating the process of norm formation have focused their analysis on an episode level. This means that analysis is focused on a relatively small amount of classroom discourse, sometimes as little as fifteen or twenty minutes total (Cobb & Whitenack, 1996). Because of its narrow focus, an episode-level analysis can go into great detail in studying the particulars of a teacher’s discursive moves and their immediate effect. However, the main limitation of an episodic analysis is its limited context: it analyzes the chosen episodes in isolation from the rest of the school year. For example, assume that a ten
minute classroom episode is analyzed and a norm is found to emerge. Did the norm emerge solely because of what happened during those ten minutes, or were there crucial episodes the previous day that helped prepare the way for the norm to emerge? Now that the norm has emerged, is it firmly established or will follow-up conversations be necessary in the upcoming days in order to establish the new norm? These are questions that an episodic analysis cannot answer because they require a wider focus and a larger data set. A longitudinal analysis analyzes data over time. It may include episodic analysis but widens its focus to consider the relationships between episodes. Because of its larger data set, a longitudinal analysis allows the researcher to make more encompassing assertions (Cobb & Whitenack, 1996). My study intends to use longitudinal data to analyze the process of norm formation.

This study is designed to answer the following research questions:

1. **What social and sociomathematical norms are associated with mathematically productive discourse?**

2. **What strategies can a teacher use to establish these norms in their classroom?**
Chapter 2: Review of the Literature

This chapter will review the research literature relevant to the two central research questions presented at the end of the previous chapter. These main research questions are:

1. What social and sociomathematical norms are associated with mathematically productive discourse?
2. What strategies can a teacher use to establish these norms in their classroom?

The literature review will contain three main foci: the motivation and research supporting the definition of mathematically productive discourse, norms associated with this type of discourse, and the process of establishing new classroom norms.

Motivation, definition, and rationale for productive discourse

Researchers over the past several decades, such as Silver and Smith (1996), Hufferd-Ackles, Fuson, and Sherin (2004), and Williams and Baxter (2010), have traced interest in mathematical discourse back to the 1989 National Council of Teachers of Mathematics (NCTM) Curriculum and Evaluation Standards for School Mathematics (Standards) and subsequent 1991 Professional Standards for Teaching Mathematics (Professional Standards). The NCTM’s discursive vision stood in contrast to the prevailing form of mathematical discourse at the time. Mehan (1979) explains this prevailing form of discourse as I-R-E: initiation, response, and evaluation. The teacher initiates a question to which he or she already knows the answer, a student responds with an answer, and the teacher evaluates the correctness of the student’s response. In classrooms where the I-R-E pattern is common, students usually play a passive role (Turner, et al., 1998). Researchers have observed that the questions posed by the teacher in an I-R-E sequence usually place little cognitive demand on the student (Graesser, Gernsbacher, & Goldman, 2003), and often result in low student participation in classroom discourse and activities (Brown & Palincsar, 1986). Because the I-R-E sequence is primarily focused on evaluating students, frequent use of it tends to
promote a performance-mentality in students (Turner et al, 2002). This mentality causes students to value correct answers over conceptual understanding. It also increases the prevalence of various face-saving strategies among students, such as avoiding difficult problems and avoiding asking for help, in an attempt to appear competent (Turner et al., 2002). The I-R-E sequence may also promote a more superficial understanding of mathematics. The evaluative part of the sequence rarely focuses on how the student arrived at their answer or why the answer is correct or incorrect (Ball, 1991). Consequently, students learn to rely on the teacher’s or textbook’s authority to judge their answers, while the criteria and rationales for these judgments remain hidden from them.

Since the NCTM reform documents valued conceptual understanding more than computational performance, they called for a concomitant shift in discursive emphasis. The NCTM made it clear that in their vision, the I-R-E discursive form, referred to as “traditional” discourse, was being replaced.

A second feature of the teacher's role is to be active in a different way from that in traditional classroom discourse. Instead of doing virtually all the talking, modeling, and explaining themselves, teachers must encourage and expect students to do so. Teachers must do more listening, students more reasoning. (NCTM, 1991, p. 36)

Unfortunately for the world of mathematics education nomenclature, the NCTM never provided a concise term to encapsulate this new discourse. Consequently, a plethora of new terms have been coined by researchers over the years as they have attempted fill this terminological void. Examples of some of these terms include the following: discourse-oriented teaching (Williams & Baxter, 1996), discourse community (Sherin, 2002), math-talk learning community (Hufferd-Ackles, Fuson, Sherin, 2004), accountable talk (Michaels, O’Connor, Resnick, 2008), high-press discourse (Kazemi & Stipek, 2001), and productive discourse (Nathan & Knuth, 2003; McClain & Cobb, 2001). As explained in chapter 1, I have chosen the term *mathematically productive discourse* (or simply, *productive discourse*) and have defined it to be discourse that has three qualities: mathematics as authority, sense-making, and mathematical coherency. The reader should observe at this point that there is not a sharp distinction between these three qualities. Making sense of a problem might come as a result of recognizing
connections with other areas of mathematics. Or attaining greater coherency might result from making sense of a problem. Appealing to a mathematical basis might help make sense of a situation. Interrelations abound between the three attributes. These examples were given merely to recognize this fact and not to exhaustively catalogue all of the interrelations.

Each of the three defining qualities of productive discourse has been recognized to be of great importance to learning mathematics. I will begin by discussing the importance of letting mathematical reasoning be the authority for evaluating work. The authority of mathematical reasoning in the broader mathematics community is witnessed by the importance given to proofs and proving. The NCTM, in the 2000 *Principles and Standards for School Mathematics*, highlighted reasoning and proof as one of its standards and stated that by the end of their K-12 education, all students “should be able to understand and produce mathematical proofs” (p. 56). The newer Common Core State Standards for Mathematics (CCSSM) contain the construction of viable arguments by appealing to mathematical reasoning as a practice standard. “Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures” (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA], 2010, p. 6). The National Research Council (2000) publication, *How People Learn*, recognizes that the deductive reasoning of proofs is “used to settle disputes and disagreements. Answers are right because they follow from some agreed-upon assumptions through series of logical steps” (p. 151). Karunakaran (2014) points out that the importance of proof is widely recognized in the mathematics community and is considered by many to be the very essence of mathematics itself. It seems clear then that in the broader mathematics community, mathematics is the authority for evaluating work. Ellis (2007) believes that justification in the lower grades using mathematical reasoning as the authority may ease the transition to more formal proofs in later grades. So by letting mathematics be the authority, whether in formal proofs or less formal arguments, teachers are allowing students to experience what many believe is the essence of mathematics. Also, as mentioned earlier, when mathematics is not the evaluative authority, the teacher tends to be.
When this happens, the evaluative criteria are rarely explicated, leaving students unsure of why their answer was correct or incorrect (Ball, 1991). This situation restricts both students’ autonomy and depth of understanding. In contrast, when mathematics is made the authority, students can understand why an answer is correct or incorrect. They can then evaluate answers for themselves, gaining greater autonomy and understanding in the process.

Like the authority of mathematical reasoning, sense-making has also been recognized as an important goal in mathematics. The National Research Council (2000) recognizes that sense-making is a natural process for human beings that begins at a very young age. Teaching sense-making helps to develop metacognition, an explicit awareness of one’s own thinking and level of understanding (National Research Council [NRC], 2000). This in turn has been shown to increase the degree to which students can transfer their learning to new contexts. An awareness of one’s own understanding, what makes sense and what doesn’t, is characteristic of an expert’s thinking in a particular area (Hatano & Inagaki, 1986). The recognition of the importance of sense-making is also reflected in the Common Core Standards. Sense-making is the first practice standard. “Mathematically proficient students start by explaining to themselves the meaning of a problem…” (NGA, 2010, p. 6).

Mathematical understanding is an often-used term in the math education community. However, it is somewhat of a vague term. What does “understanding” entail? I use the following explanation:

Conceptual understanding refers to an integrated and functional grasp of mathematical ideas. Students with conceptual understanding know more than isolated facts and methods. They understand why a mathematical idea is important and the kinds of contexts in which it is useful. They have organized their knowledge into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know. (NRC, 2001, p. 118)

This explanation defines understanding in terms of coherency, how ideas relate to other ideas. “Things take meaning from the ways they are related to other things” (Fennema & Romberg, 1999, p. 20). Rather than use the term understanding, I have opted to use coherency because of its greater clarity and
precision. So why is coherency important? Much research indicates that we learn new ideas by connecting them to those we already know (NRC, 2001). This means that the human mind naturally seeks coherency. Coherency increases the usability of knowledge. It increases the likelihood that an individual will be able to transfer what he or she already knows to a new situation (NRC, 2000). Given a particular subject, an expert’s knowledge differs from a novice’s knowledge not only because it is more extensive but also because it is more coherent. Experts’ knowledge is organized and structured around important concepts, allowing them “to see patterns, relationships, or discrepancies that are not apparent to novices” (NRC, 2000, p. 17). Experts tend to remember information more easily in their area of expertise not because their memories are inherently superior, but because their richly structured framework aids storage and recall of information (NRC, 2000). Without a coherent mental framework, knowledge is learned more slowly, forgotten more easily, and does not transfer to new situations (Fennema & Romberg, 1999).

Extremely isolated knowledge, say of an algorithm, may lead a student to believe that even a slightly different problem requires a completely different approach. The 2011 Trends in International Mathematics and Science Study (TIMSS) found that students from Japan, China, and Hong Kong outperformed their American counterparts on mathematics achievement tests (International Association for the Evaluation of Educational Achievement, 2012). International studies comparing pedagogy across nations have found that Japanese, Chinese, and Hong Kong teachers tend to place more emphasis on mathematical coherency in their lessons than American teachers, which may partially explain the higher performance of these East Asian countries on the TIMSS assessments (Corey, Peterson, Lewis, & Burkarau, 2010; Bryan, Wang, Perry, Wong, & Cai, 2007). It is crucial then that students attain mathematical coherency. The goal of coherency, phrased in terms of making connections, is a process standard in the 2000 Principles and Standards for School Mathematics. The seventh Common Core practice standard also deals with coherency, using the language of mathematical structure.

The prior explanations provide some of the rationale for my choice of defining qualities for productive discourse. But a careful reading of the Standards and Professional Standards will reveal that the NCTM’s envisioned discourse has many other qualities ascribed to it. It may reasonably be asked:
why include sense-making, coherency, and mathematics as authority as the defining qualities of productive discourse? Why not other qualities as well? As explained above, each of the three qualities was chosen because research indicates it has a significant positive impact on learning. So these three qualities have each been shown to positively affect the outcome, or product, of discourse, hence the label productive discourse. This is not necessarily true for many of the other discursive qualities listed in the Standards and Professional Standards. For example, the Professional Standards also emphasizes active student participation in discourse, stating that “teachers must do more listening, students more reasoning” (NCTM, 1991, p. 36). However, less teacher and greater student participation in discourse does not necessarily lead to increased learning. Nathan and Knuth (2003) observed a middle school teacher who specifically attempted to increase the frequency of student participation in discourse while simultaneously reducing her own. She was successful in accomplishing her goal, but the resulting discourse often had “a lack of rigorous argumentation and evidence.... Student ideas were offered publicly for others to pick up, refute, or ignore, often with no basis for evaluation other than opinion” (Nathan & Knuth, 2003, pp. 199-200). The Professional Standards also stresses that students’ thinking should be respected. Certainly, this is an admirable quality by any ordinary standard, but it also does not necessarily lead to increased student learning. In the classroom that Nathan and Knuth (2003) observed, one of the teacher’s main goals was for students to value and respect each other’s thinking. She stressed this to her class so frequently that the researchers referred to it as her “general tenet” (Nathan & Knuth, 2003, p. 186). Nevertheless, as the earlier quote from Nathan and Knuth (2003) indicates, the resulting discourse frequently did not aid student learning. Many other discursive qualities mentioned in the Standards and Professional Standards could be used as examples that do not necessarily lead to student learning. The common trait shared by most of these examples is that they relate to the form of discourse rather than its substance. This idea is articulated well by Lobato, Clarke, and Ellis (2005). These researchers point out that discursive moves with the same form (e.g. questioning, providing information) could have many different intentions behind them and lead to many different outcomes depending on contextual factors. For example, consider the discursive quality of frequent teacher questioning. These teacher questions could potentially lead students
to connect their claims with a mathematical basis. They could also potentially be evaluative questions that require a one-word response. For this reason, the defining qualities of productive discourse should not be form related.

At this point, it might still be reasonably asked: aren’t there other discursive qualities that positively affect student learning? Why aren’t these included as definitional? Firstly, there are indeed additional qualities that positively affect student learning. However, many are substantially interconnected with the qualities that I have already chosen as definitional. For example, one might suggest that frequent justification, generalization, or argumentation should be included as part of the definition of productive discourse. But what makes a justification or argument beneficial to learning? I argue that appealing to a mathematical basis (whether to validate or refute) is what makes such an activity beneficial to learning. A justification that appeals to the teacher as an authority has many of the same shortcomings as the I-R-E form of discourse mentioned earlier. Likewise, a generalization is an extension of one’s coherency, for it involves connecting ideas across contexts.

All of the Common Core practice standards are interconnected with the three qualities that I have chosen. The first Common Core practice standard (CCSSM 1) deals explicitly with sense-making. Regarding the other practice standards, I make the following contentions: CCSSM 2 (reasoning abstractly) and CCSSM 3 (constructing viable arguments) involve a combination of sense-making, appealing to mathematical bases, and creating coherency. CCSSM 4 (modeling) and CCSSM 5 (using appropriate tools strategically) require sense-making. CCSSM 6 (attending to precision) involves appealing to mathematical bases. CCSSM 7 (mathematical structure) and CCSSM 8 (regularity in repeated reasoning) involve creating coherency. In asserting that all of the practice standards involve the defining qualities of productive discourse, I do not assert that they only involve these three qualities. Certainly, they also involve other qualities. However, the interrelation of my definition of productive discourse with all of the Common Core practice standards is one possible indication that my definition is not lacking anything substantial. It is not an absolute indication, just as the Common Core is not absolute. Using interrelated qualities, my definition could no doubt be worded differently. I could also probably
add additional qualities as well. However, a lengthy definition is not necessarily superior or more useful
than a shorter one. In the interest of parsimony, I prefer not to add additional interrelated qualities to my
definition of productive discourse, finding it sufficient for the purposes at hand.

The goal of my first research question is to investigate which norms are associated with
mathematically productive discourse. However, productive discourse itself is defined by three particular
qualities. These definitional qualities describe regularities in the discourse. Recall though that norms are
constructs that describe regularities in classroom social behavior. Therefore, the term *mathematically
productive discourse* is actually defined by three particular *norms*: the norm of holding mathematics as
authority, the norm of sense-making, and the norm of striving for mathematical coherency. These three
definitional norms are fairly abstract; there are many specific ways that a class could hold mathematics as
the authority or strive for mathematical coherency. The goal of my first research question is to identify
more specific, concrete norms that help support these three definitional norms. This goal may be
visualized by Figure 2-1 below. The norms farther up in the hierarchy represent more abstract, general
norms. As one moves lower down in the hierarchy, the supporting norms become more specific and
concrete. As an example, consider sense-making. A supporting norm for this might be exploring incorrect
reasoning. Rather than immediately dismissing student mistakes as incorrect, the class investigates *why*
the reasoning is incorrect. Below “exploring incorrect reasoning” on the hierarchy may be an additional
supporting norm: emphasizing understanding over performance. Getting the correct answer as quickly as
possible is not as important as developing understanding. So in this example, the norm of “emphasizing
understanding over performance” supports the norm of “exploring incorrect reasoning” which in turn
supports the norm of sense-making. This is merely one example of a possible relationship that might
emerge between norms. Research by McClain and Cobb (2001) supports the notion that the establishment
of certain norms might then allow the establishment of others, leading to a hierarchical relationship
between norms.
In reviewing the research literature regarding productive discourse, one particular language issue must be clarified first. As just explained, productive discourse is a term that I have chosen and defined specifically. Given the plethora of competing discursive terminology in math education, one might wonder if another term is really necessary. I believe that it is for several reasons. First, many of the competing discursive terms are quite vague. For example, Hufferd-Ackles, Fuson, and Sherin (2004) discuss the development of a math-talk learning community. They define this term as “a classroom community in which the teacher and students use discourse to support the mathematical learning of all participants” (p. 82). However, no specific qualities of “mathematical learning” are explicated, nor are the ways in which the teacher and students “use” discourse. If “mathematical learning” is defined as procedural speed and accuracy, then the I-R-E form of discourse could be said to support mathematical learning, making a traditional mathematics classroom a math-talk learning community. From the context of their paper, it is clear that Hufferd-Ackles, Fuson, and Sherin do not actually think of “mathematical learning” in this sense. Nevertheless, this does not change the vagueness of their definition. As another example, Sherin (2002) defines a discourse community as a classroom community where “students are expected to state and explain their ideas and to respond to the ideas of their classmates” (p. 207). But how
are students expected to respond to the ideas of their classmates? Would grunts of acknowledgement suffice as “responding” to the ideas of peers? Again, from the context of the paper, it is clear that Sherin has some definite criteria in mind, but this is not made explicit in her definition of discourse community. In addition to the issue of vagueness, many researchers have focused on discursive features that are not necessarily associated with student learning. An example of this has already been provided by Nathan and Knuth (2003). By defining a new term, I can not only be more precise, but also focus strictly on norms that have a demonstrated correlation with student learning. The term productive underscores this correlation with student learning, which is itself the product of education. Given this difference in discursive terms and definitions, interpretation is necessarily when reviewing the research literature. Rather than focusing on specific terms that other researchers use, I will instead look for evidence (or lack thereof) of the defining norms of productive discourse.

**Norms associated with productive discourse**

In this next section, I provide an overview of norms known to be associated with productive discourse. One such norm is that students draw comparisons between different solution strategies. When students pick one solution strategy out of multiple possibilities and present it to the class, they often must defend why they made the choices that they did (Stein, 2001). This tends to promote the authority of mathematical reasoning as they appeal to a mathematical basis to justify their decisions. To compare solution strategies, students must first make sense of the mathematics in their peers’ approach (Hufferd-Ackles, Fuson, & Sherin, 2004). The act of comparing promotes coherency as students understand other strategies in terms of similarities and differences to their own. Making these connections grants students greater autonomy as they take on more active discursive roles such as critic, helper, defender, and supporter. It is imperative to note that the act of comparing solution strategies is what is conducive to productive discourse; the mere presence of multiple solution strategies is not necessarily productive. Kazemi and Stipek (2001) observed two fourth-grade classes where “sharing strategies during whole-class discussions looked like a string of presentations, each one followed by applause and praise” (p. 72).
Because no comparisons were made, Kazemi and Stipek noted that there was a low-press for conceptual understanding. By contrast, the classrooms that emphasized conceptual understanding frequently compared strategies by using mathematical criteria. In the middle school classroom that Nathan and Knuth (2003) observed, students freely shared their solution strategies, but since no comparisons were drawn between solutions, the discourse was aimless. “Student ideas were offered publicly for others to pick up, refute, or ignore, often with no basis for evaluation other than opinion” (Nathan & Knuth, 2003, pp. 199-200).

A comparison (of any type) between multiple objects presupposes a standard of some sort by which to judge similarities and differences in the objects. A comparison then between mathematical solution strategies requires a mathematical standard (basis). In some cases, it may be necessary to explicitly discuss the basis for appropriate sharing of solutions. McClain and Cobb (2001) observed a first-grade teacher who frequently tried to elicit multiple solution strategies from her students. However, when she asked for different solutions, students frequently shared repetitious ideas. The result was “a series of disjoint turns in which students often… offered explanations that did not contribute to the mathematical agenda…. There in fact was often little of mathematical significance to attend to in the discussions” (McClain & Cobb, 2001, p. 247). Because the teacher had not explicated her criteria for different, students were left to guess at what she meant. This issue is reminiscent of the I-R-E form of discourse. In both situations, the teacher has certain criteria in mind, but fails to explicate them. This limits the productivity of discourse because students cannot relate their ideas to a mathematical basis, but must instead rely on authoritative teacher evaluations without understanding the rationale behind them. In McClain and Cobb’s (2001) classroom, the teacher eventually began to explicate her criteria for difference. Once students understood the criteria, they promptly began appealing to this newly established mathematical basis to justify why their solution was (or wasn’t) different from others. Hence, one of the factors in this situation limiting discursive productivity was the lack of a clearly developed standards.

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4Part of Kazemi and Stipek’s (2001) definition of “high-press” is the “development of better understandings” (p.61). Context indicates that they seem to use this term to mean sense-making.
sociomathematical norm for difference. Students simply did not know what criteria to use in considering solution strategies.

The expectation that students “explain their thinking” is a common social norm in classrooms attempting to implement the NCTM’s vision of reform (Kazemi & Stipek, 2001). Research indicates however that procedurally focused explanations are not necessarily associated with productive discourse. As Kazemi and Stipek (2001) point out, “students can describe the steps they took to solve a problem without explaining why the solution works mathematically” (p. 64). As an example, consider the college classroom that Clement (1997) observed. The teacher frequently commented in interviews on the importance of students explaining their work, and Clement’s observations confirmed that students did regularly explain their work. However, such explanations usually consisted of summarizing the steps taken to arrive at the answer with no explanation of the rationale for choosing those steps. After the final exam, an interview was conducted with one of the highest-performing students from the class about one of the exam questions. Although the student answered the exam question correctly, he was unable to justify his procedural steps using mathematical reasoning. Instead, he selected a previously known procedure that he believed had an appropriate level of complexity for a final exam question (Clement, 1997). This indicates that mere discussion of procedure does not necessarily promote sense-making or the math as authority. Kazemi and Stipek (2001) observed that the norm of explaining the rationale behind procedural steps was a distinguishing factor in how much a classroom pressed for conceptual understanding. “In high-press exchanges, students went beyond descriptions or summaries of steps to solve a problem; they linked their problem-solving strategies to mathematical reasons. In whole-class settings, teachers expected presentations of strategies to include both explanation and justification” (p. 64). This norm is productive for a number of reasons. As students explain the reasons behind their steps, they make sense of the procedure; that is, the procedure takes on meaning within the context of the problem. As students justify their computational choices, mathematical reasoning becomes the authority. The procedure itself is no longer an authority and neither is the teacher nor textbook that taught the procedure, but the reasoning behind the procedure. Furthermore, when students justify their steps, they
tend to relate verbal, graphical, and numerical strategies (Kazemi & Stipek, 2001). In doing this, students develop coherency as they connect the procedure to specific contexts and other areas of mathematics. No longer is the procedure an isolated piece of information in the students’ minds.

Students naturally tend to conflate mathematical criticism of their ideas with personal criticism of themselves. This is likely because in everyday life, students are not used to drawing a sharp distinction between themselves and their ideas (Lampert, Rittenhouse, and Crumbaugh, 1996). When their ideas are criticized, students often feel personally attacked, and their corresponding defense may have more to do with social status and saving face rather than mathematics. As mentioned earlier in chapter 1, Lampert, Rittenhouse, and Crumbaugh (1996) provided examples of students making arguments for their ideas based on nonmathematical factors such as penmanship, social status, feelings of confidence, and rhetorical moves. Discourse such as this is unproductive because these factors become the authority in place of mathematical reasoning. Furthermore, when students are jockeying to save face in front of their peers, they are unlikely to be focused on making sense of the mathematical content. This confusion of mathematical and personal criticism not only stifles productive discourse then, but also discourse in general. When students view contributing ideas as a potential risk of looking incompetent, they become reluctant to contribute anything out of fear of embarrassment (Wood, Cobb, & Yackel, 1991). Productive discourse then requires a social norm of differentiating between mathematical criticism of ideas and personal criticism of themselves. As an example, Wood (1999) observed a second-grade teacher who began explicitly emphasizing from the first day of school the difference between personal and mathematical criticism. She stressed that when her students disagreed with each other, they needed to provide a reason for the disagreement. As the school year progressed, Wood noted that these patterns for disagreement became normative and concluded that the teacher successfully established an environment where students were “able to experience mathematics as a discipline that relies on reasoning for the validation of ideas” (p. 189). This separation of personal and mathematical criticism helped in establishing mathematical reasoning as the authority in the classroom.
Research indicates that the frequency of productive discourse may fluctuate depending on the position of a class within a topic. When the class begins a new topic, the teacher tends to be more directive and dominant in the discourse in order to introduce new terms and explain new concepts (Hufferd-Ackles, Fuson, & Sherin, 2004; Smith, 2000). After students gain some initial familiarity with the new terminology, their involvement in the discourse is able to increase as they begin to make sense of the new concepts and understand them in terms of their current knowledge. Mendez (1998) explains this phenomenon, claiming that in order for students to participate meaningfully in discourse, the mathematical content must be in their zone of proximal development. In other words, the students must have the right amount of familiarity with the content in order to engage with it. On one hand, if students are overly familiar with a topic, then discourse about it will likely not lead to any further sense-making or coherency. But on the other hand, if students have no familiarity at all with the content, they will be unable to contribute any ideas about it. They may even struggle to formulate questions. Research indicates that when an individual knows nothing about a topic, they often do not know what to ask about it (Cohen, 1991). In a case such as this, the individual is not even aware of concepts they could potentially inquire about. McClain and Cobb (2001) also observed that productive discourse temporarily declined during the beginning of a new topic. Students in their first-grade classroom were used to appealing to mathematics as the authority. However, at the very beginning of a new topic, students had to learn a certain amount of new mathematical criteria before they could begin appealing to mathematics as the authority within the new topic.

**The process of building norms**

Recall the research questions:

1. *What social and sociomathematical norms are associated with mathematically productive discourse?*

2. *What strategies can a teacher use to establish these norms in their classroom?*
Note that the first question deals with identifying norms that possess a certain quality: association with productive discourse. The second question deals with teacher strategies used during the process of establishing these norms. Before considering the process of establishing productive norms, I first discuss why this process needs explicit focus. At this point, it may be helpful to recall the relationship between beliefs and norms. Norms are the collective correlate of beliefs. Individuals have beliefs while groups (considered as a whole) have norms. For consistency, I will always use norms for groups and beliefs for individuals. Research indicates that many of the beliefs that students tend to bring into the classroom are not conducive to productive discourse. This was already discussed in part earlier when it was noted that students naturally tend to conflate mathematical criticism with personal criticism (Lampert, Rittenhouse, & Crumbaugh, 1996). However, the process of trying to change unproductive norms can require substantial effort on the teacher’s part. Hufferd-Ackles, Fuson, and Sherin (2004) observed a third-grade classroom where students began the school year accustomed to providing unjustified answers. Shortly into the school year, the teacher began making a deliberate effort to prompt students to share their process of reasoning along with their answer. These initial attempts were “laborious” (p. 98) and required great patience from the teacher as she often had to repeatedly prompt the students for each part of their reasoning. Other research indicates that students may actually resist the teacher if he or she attempts to change the classroom norms too abruptly. Smith (2000) provided an example of a ninth-grade teacher whose students were accustomed to math problems that required following a known procedure. When the teacher gave the students a non-routine problem, they quickly grew frustrated, gave up, and began pressing the teacher to provide more guidance. It is likely that these students had established beliefs about what it meant to “do math”, namely, following procedures. In presenting a non-routine problem, the teacher was introducing a situation that conflicted with these beliefs. Unsurprisingly, the students resisted. The presence of competing norms often leads to a variety of communicative conflicts (Sanchez & Garcia, 2014). Students may assume different criteria when justifying a solution strategy or use the same word with different understandings of its meaning (Yackel & Cobb, 1996). Or, as in Smith’s (2000) example, students may not recognize the importance or legitimacy of an activity.
Since the process of changing unproductive norms can be laborious and even face student resistance, it is important that the teacher adopts effective strategies when trying to build new, productive norms. But this raises the question: what strategies are effective in building productive norms? Case studies in the research literature have identified teachers using several different strategies to effectively establish productive norms. One such strategy is initiating normative discussions: discussions where norms are the explicit focus. Such conversations can be proactive in anticipation of an upcoming activity, or reactive, debriefing an activity or interaction that just occurred. For example, Wood (1999) observed a second-grade teacher who began initiating normative discussions on the very first day of school. The teacher focused on differentiating between personal and mathematical criticism. She clearly stated her expectations for agreement and disagreement and used hypothetical situations to concretely illustrate her points. She then initiated a mathematical discussion with her students, giving them an opportunity to interact according to these norms. During this discussion, the teacher coached her students in their interactions, giving them feedback and clarifying whether or not they had fulfilled their expected roles. This teacher’s strategy was to intentionally give her students opportunities to discuss and then immediately experience the norms she wanted to develop. After observing consistent norms during the remainder of the school year, Wood (1999) concluded that “early in the school year, the children appear to have understood the teacher’s expectations for both themselves and others” (p. 188). As another example of normative discussions, Wood, Cobb, and Yackel (1991) observed an elementary teacher leveraging certain situations that spontaneously arose to initiate normative discussions. When one particular small-group was not sharing responsibility for problem-solving, the teacher explicitly corrected them. Later in the same class session, she made that group share with the entire class about that particular issue and how they rectified it. This incident became a concrete illustration that the teacher used to initiate future normative discussions for small-group work. Wood, Cobb, and Yackel (1991) noted that key incidents, regardless of whether they were exemplars or transgressions of norms, could become “paradigm cases” (p. 398) that the teacher could then use as references in subsequent normative discussions. They also noted that “as the school year progressed, the children became increasingly adept
at creating productive relationships without the teacher’s assistance” (p. 400), thus indicating that the teacher was successful in achieving her original normative goals. Staples (2007) observed a ninth-grade teacher over the course of a year who regularly employed normative discussions to help her students understand both how and why the class engaged in particular mathematical practices. Reflecting on the school year, Staples noted that “in the fall, it was a regular occurrence for students to make comments during class time that demonstrated views that were more likely to impede than enhance collaborative work, such instances were uncommon in the spring” (p. 209). Other research indicates that normative discussions may be not only helpful but necessary if teachers are to successfully establish their normative goals. Levenson, Tirosh, and Tsamir (2009) interviewed two elementary teachers about their beliefs regarding how mathematical explanations should be given. Subsequent observation of the two classrooms revealed that neither teacher initiated normative discussions about the nature of mathematical explanations. Students’ mathematical explanations varied widely and did not appear to conform to the teachers’ beliefs. Student interviews confirmed that there were no consistent beliefs among the students as to how mathematics should be explained. Hence, it appears that these teachers were not successful in establishing norms for explanation.

In addition to normative discussions, both Wood (1999) and Wood, Cobb, and Yackel (1991) observed teachers frequently making normative comments. These comments typically consisted of a few sentences and included proactive reminders, praise of a positive example, and evaluation of a negative example. Wood (1999) observed a teacher using normative comments to reinforce norms introduced earlier in normative discussions. The frequency of these normative comments then decreased as the desired norms were established. Wood, Cobb, and Yackel (1991) observed their teacher using normative comments to clarify how a particular norm, introduced earlier via normative discussion, applied in a new situation.

Research indicates that the emergence of certain norms may then enable the subsequent emergence of others, i.e. certain norms seem to be pre-requisites for others. In McClain and Cobb’s (2001) first-grade classroom, the sociomathematical norm of mathematical difference emerged about one
month into the school year. This norm consisted in part of criteria that specified what constituted a different solution. Shortly after this norm emerged, other sociomathematical norms emerged, such as norms regarding easy solutions and efficient solutions. McClain and Cobb noted that the emergence of the difference norm served as a basis for the emergence of the other norms. Once students had criteria to differentiate between solutions, they could then develop criteria to further differentiate how the solutions differed.

The emergence of new norms can sometimes conflict with existing norms. For example, in Wood, Cobb, and Yackel’s (1991) second-grade classroom, one well-established norm was the expectation that students make sense of a problem for themselves rather than relying on a peer. However, a second norm was the expectation that students share their thinking with their peers. Conflicts arose because students were unsure of how these two norms were to coexist. In one instance, a student was trying to share his thinking with a partner, but the partner was still engaged in individual sense-making and was ignoring his friend. The partners were each trying to fulfill a different norm, and it was unclear which norm took priority over the other. To successfully establish new norms then, the teacher might have to explicitly clarify how new norms function in relation to preexisting ones (Wood, Cobb, & Yackel, 1991).

**Episodic and longitudinal analysis: insights they offer**

Most studies investigating the process of norm formation have focused their analysis on an episode level. An episode is a subsection of classroom discourse with a distinguishable beginning and end, centered on a particular question or issue. As is often the case with terminology, there is an inherent amount of imprecision in this term. What exactly constitutes the beginning and end of a particular issue? Given a transcript of classroom dialogue, there will be many ways to legitimately define episodes within it. The point here is not to argue that there is only one way to define episodes, but rather that the definition of episodes will depend on what exactly is being analyzed. The precise focus of the analysis will determine exactly how much dialogue should be grouped together for consideration. As a general
rule of thumb though, Hufferd-Ackles, Fuson, and Sherin (2004) believe most typical classroom episodes will range from five to ten minutes in length. Although this is merely one opinion, it provides a helpful guideline to consider when defining episodes. An episodic analysis of norms typically scrutinizes a small number of episodes that have contributed to the formation of a particular norm of interest. Because of its focus on a relatively small amount of dialogue, an episodic analysis can go into great detail in studying the particulars of a teacher’s discursive moves and their immediate effect. However, the main limitation of an episodic analysis is its limited context: it analyzes the chosen episodes in isolation from the rest of the school year. For example, assume that a ten minute classroom episode is analyzed and a norm is found to emerge. Did the norm emerge solely because of what happened during those ten minutes, or were there crucial episodes the previous day that helped prepare the way for the norm to emerge? Now that the norm has emerged, is it firmly established or will follow-up conversations be necessary in the upcoming days in order to establish the new norm? These are questions that an episodic analysis cannot answer because they require a wider lens and a larger data set. A longitudinal analysis analyzes data over time. It may include episodic analysis but widens its lens to consider the relationships between episodes. Because of its larger data set, a longitudinal analysis allows the researcher to make more encompassing assertions (Cobb & Whitenack, 1996).

Longitudinal analyses of classrooms have revealed several important trends in the process of norm development. First, most of the major norm development seems to occur during the first two months of the school year. Norms tend to remain stable for the remainder of the school year with only minor adjustments occurring (McClain & Cobb, 2001; Wood, 1999; Wood, Cobb, & Yackel, 1991). Second, teachers seem to begin the school year using concrete situations and illustrations to discuss norms. Some of these situations may become cases that the teacher can refer back to later. As norms become established, further discussion of them can become more general and abstract. For example, in Wood, Cobb, and Yackel’s (1991) second-grade classroom, the teacher initially discussed the norm of collaboration in specific detail with her students: they should all be actively participating, sharing their thinking, ensuring all group members understand before moving on, etc. These expectations were often
discussed in relation to concrete situations that had just occurred. By midway through the year however, the teacher would simply remind the students to share and cooperate in their groups. Because the specific details of cooperation had already been established, the teacher could simply remind the students generally to cooperate. Third, teachers seem to use normative discussions to build norms and then normative comments to maintain them. In Wood’s (1999) elementary classroom, the teacher frequently initiated normative discussions during the first few weeks of the school year. During the next month, she relied mainly on regular normative comments to maintain and reinforce these norms. By two months into the school year, the frequency of normative comments had declined and almost disappeared as the desired norms became established. Finally, teachers may introduce norms gradually at the beginning of the school year. In Wood’s (1999) classroom, the teacher spent the first week emphasizing that students needed to listen actively to their peers and indicate agreement or disagreement with claims. During the second week, she extended these initial norms by adding that students needed to give reasons for their disagreement, and ask clarifying questions if they didn’t understand their peer’s claim. In McClain and Cobb’s (2001) first-grade classroom, new norms emerged after a pre-requisite norm had been established.

In reviewing the relevant research literature, this chapter focused on several key areas. The first was the research motivating and ungirding the definitional norms for mathematically productive discourse. Each of the three definitional norms was selected because of its beneficial impact on learning. These are by no means the only norms beneficial for learning, but for my purposes they are sufficient to create a definition for mathematically productive discourse. These norms were contrasted with form-related discursive qualities, which do not necessarily lead to increased student learning. After establishing the definition of productive discourse, this chapter then surveyed norms that have been associated with this kind of discourse. Finally, the chapter discussed what the literature says about the process of intentionally building classroom norms. The difficulties of establishing new norms were discussed, as well as strategies that were effective in several case studies. The potential insights that longitudinal and episodic analyses offer were also discussed as well. This study utilizes a longitudinal analysis, which will be discussed in the next chapter on methodology.
Chapter 3: Methodology

As discussed in the literature review, while much is known about both social and sociomathematical norms, most of this knowledge comes from analysis focused on episodes. Such episodic analysis may focus entirely on only a few minutes of classroom discourse. Less is known about the long-term process of developing norms. Investigating this process requires a longitudinal study so that episodes may be analyzed in terms of their precursors and subsequent impact on future episodes. Few longitudinal studies have focused on the process of developing norms. Two longitudinal studies, Wood (1999) and Wood, Cobb, and Yackel (1991), have focused strictly on social norms, while another longitudinal study, McClain and Cobb (2001), focused on the development of one particular sociomathematical norm. A longitudinal study by Staples (2007) focused on the development of sociomathematical norms that supported the generation of ideas on a group-level rather than an individual level.5 Recall the remark made by Kazemi and Stipek (2001):

> Although we propose that at least four sociomathematical norms worked together to create a press for conceptual learning, continued research may reveal other norms that contribute to a high press. It is also important to investigate, with longitudinal data, how sociomathematical norms are created and sustained, and how they influence students' mathematical understanding. (p. 79)

The goals of this study are guided by the research questions:

1. **What social and sociomathematical norms are associated with mathematically productive discourse?**

2. **What strategies can a teacher use to establish these norms in their classroom?**

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5 Using the term “collaborative inquiry,” Staples explains that she focuses specifically on discourse where mathematical ideas originate and develop on a group level, rather than on an individual level. My definition of “mathematically productive discourse” does not place any restrictions on the origin of ideas. They may originate on a collective level, or on an individual level and then be shared with the collective.
However, norms cannot be investigated in an abstract manner devoid of context. They are inherently tied to the community in which they arose. For this reason, a case study research design is an appropriate choice for my study. A case study explores a larger issue of interest by performing an in-depth investigation of a particular instance, called the case, where the larger issue manifests itself. The case has been defined by Merriam (1998) as “a thing, a single entity, a unit around which there are boundaries” (p. 27). Miles and Huberman (1994) define the case as “a phenomenon of some sort occurring in a bounded context” (p. 25). The actual case itself could consist of a person, a program, a group, a school, and so on. What actually comprises the case is less important than the fact that it is naturally bounded in some way so that it can be clearly delineated. In my situation, the larger issues of interest are norms associated with productive discourse and teacher strategies used to establish them. The case would be a classroom where productive discourse occurs. In my situation, the case is easily defined since a classroom is a naturally bounded entity. It is important to stress again that although the case is the immediate focus of the study, it is not the object of ultimate interest. The case is studied merely because it illustrates the larger issue, which is itself the ultimate topic of interest. Studying the case is a means of studying the larger issue. So in my situation, this means that I will study a particular classroom in order to contribute to a larger body of knowledge about norms and how teachers develop them.

Case studies typically involve a longitudinal, in-depth study of the case (Creswell, 2012) and are often concerned more with a process than an outcome (Merriam, 1998). These qualities of a case study make it ideally suited to investigate my research questions. Identifying the norms of a community requires an in-depth investigation. However, regularities cannot be observed from a quick or casual observation. My study is also concerned with the process of norm development rather than just the outcome. Studying this process requires longitudinal data, which means many in-depth observations over time. Therefore, the in-depth nature of the case study method fits well with the research questions I am investigating. Also, many in-depth observations will help support the general validity of my findings by allowing me to become well-acquainted with the case study classroom and draw on a larger collection of data (Merriam, 1998). At this point, it might reasonably be asked: why limit the study to a single case? Why not study
multiple classrooms? Certainly, this would be ideal. Multiple cases would further strengthen the validity of the results and possibly increase the generalizability of them. However, given the substantial time requirements of performing a longitudinal, in-depth investigation of just a single class, a multiple case study will not be pursued.

In the remainder of this chapter, I will describe the classroom I have chosen to study and my rationale for selecting it. I will then proceed to describe the two phases of my study, which focus on answering the two respective research questions. For each phase, I will outline the processes of data collection and data analysis.

**Selection and description of the case**

Since I am employing a single case study approach, the selection of an appropriate case is crucial. Case study researchers employ *purposeful sampling*, strategically choosing a case “from which one can learn a great deal about issues of central importance to the purpose of the research” (Patton, 1990, p. 169). Recall that in a case study approach, the case is chosen not for its own sake but because it helps illuminate a larger issue of interest. In my study, the larger issues of interest are norms associated with mathematically productive discourse and the teacher strategies used to establish them. In order to learn a great deal about this, or even to learn *anything* about this, I must select a case, a classroom, where mathematically productive discourse regularly occurs. The more frequently such productive discourse occurs, the more potential insight the case offers.

The case I have selected is a teacher who has a very particular set of values about how mathematics is done in her classroom. Ms. Skywalker has been teaching math at the elementary level for 14 years in a small college town in the northwestern United States. For the past three years, Ms. Skywalker has participated in the *Making Mathematical Reasoning Explicit (MMRE)* project, which seeks to increase the frequency of student-voiced mathematical generalizations and justifications in the classroom. To accomplish this goal, *MMRE* partnered with 35 rural school districts across two states and

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6 *MMRE* is funded by the National Science Foundation. NSF DUE #1050397
led professional development for over 70 participating teachers in both elementary and secondary grade levels. Participating teachers received professional development for a total of three years during both multi-week summer institutes and periodic meetings throughout the school year. *MMRE* professional development focused on deepening teachers’ mathematical content knowledge, helping them to recognize key mathematical concepts in a lesson and press for generalizations or justifications from their students. *MMRE* project staff periodically observed teachers’ classrooms to document if the frequency of student-voiced generalizations and justifications was increasing. It was during one of these routine observations, near the end of the 2013–2014 school year, that I first observed Ms. Skywalker’s fifth-grade classroom. I immediately noticed that students were offering unprompted justifications that appealed to a mathematical basis. For example, during routine daily work, students were investigating properties of the number 15. One student stated: “[fifteen is] odd because it’s not evenly divisible by two.” Another student stated: “[fifteen] is composite because its factors are 1, 3, 5, and 15, and not just 1 and 15.” These unprompted appeals to mathematical reasons and definitions provided compelling evidence that a norm of mathematics-as-authority existed in Ms. Skywalker’s class. This visit prompted me to approach Ms. Skywalker and inquire if she would be willing to let me study her classroom during the upcoming school year, to which she enthusiastically agreed. From my observations at the beginning of the 2014–2015 school year, I saw evidence that all three of the defining norms of mathematically productive discourse were being established. Students appealed to mathematical reasons when justifying mathematical claims or questioning another student’s work. Students regularly drew visual representations of problems to make sense of them, and used estimation to gauge the reasonableness of their answers. Students also created coherency by recognizing the interrelatedness of mathematics, such as the fact that addition and subtraction are inverse operations. In short, the discourse of Ms. Skywalker’s classroom was characterized by the defining norms of mathematically productive discourse, thus leading me to select her classroom as my case.

As observations and interviews progressed, a picture of Ms. Skywalker began to emerge. Ms. Skywalker is a high-energy teacher who brings an abundance of enthusiasm and excitement to her
classroom. She frequently laughs and makes jokes, does voice impressions and foreign accents, and often speaks with fervor. Her energy and sense of humor succeed at engaging her students and keeping the classroom atmosphere light. Ms. Skywalker also demonstrates great care for her students. One of her priorities is to build a “community of learners,” a safe environment where students feel comfortable to venture their thinking. She recognizes that sharing one’s thinking in front of peers can be intimidating because of the risk of potentially looking incompetent. Her comments to her students frequently affirm their contributions to class discussion as important and valued. Ms. Skywalker strives to quickly learn each of her student’s strengths and weaknesses, highlighting her belief that teaching is about meeting individual needs. During the first week of school, Ms. Skywalker gave her students questionnaires asking about their needs and feelings toward math. She also sent home similar questionnaires to parents about their children. She then compiled the results to create a profile for each student, helping her to better understand them.

Ms. Skywalker’s philosophy of teaching mathematics has been significantly influenced by the MMRE professional development program that she participated in. She explained to me how MMRE has shaped her perception of what mathematics fundamentally is. Prior to her MMRE involvement, she thought of mathematics as primarily about memorizing facts and formulas, but now she perceives it as more about recognizing patterns and learning reasoning strategies. Thanks to MMRE professional development, the depth of her mathematical content knowledge has significantly increased. Her mathematics teaching demonstrated a thorough understanding of the material. She often solicited multiple ideas, solutions, and representations from her students and then showed how they were related or equivalent. The MMRE program has also given Ms. Skywalker the mathematical mindset of a continual learner. “You’re never done learning math,” she explained to me. Whatever level of mathematical knowledge someone has, Ms. Skywalker is convinced they can always understand math more thoroughly. She enjoys deepening her own mathematical knowledge through her students’ various perspectives and ideas. Her MMRE involvement has made Ms. Skywalker a leader in her school district. She has helped organized and lead professional development sessions for other elementary teachers, passing on the
mathematical content knowledge and pedagogy she has learned from MMRE. In short, Ms. Skywalker is an energetic and caring teacher with a thorough command of her mathematical content, as well as a leader in her school district.

During the 2014–2015 school year, approximately 29% of the students at the case study school qualified for free or reduced lunch. Ms. Skywalker’s class consisted of 26 students. She informed me that this group of students was typical of previous classes she has taught with one exception. While in previous years, Ms. Skywalker typically had one or two “highly-capable” students, this particular class had eight. Highly-capable students are those who, after having been tested for eligibility, are regarded to be at or above the 97th percentile for academic performance. However, only two of Ms. Skywalker’s highly-capable students met this qualification for mathematics; the other six tested as highly-capable in language arts, but not mathematics.

Phase one of the project

My study consisted of two phases corresponding to the two main research questions. The first phase focused on identifying norms in Ms. Skywalker’s classroom. Since mathematically productive discourse regularly occurs in this classroom, I assumed that the norms evident during mathematics were norms associated with mathematically productive discourse. The second phase of the study focused on identifying teacher strategies used to establish the norms identified in the first phase.

Data collection for phase one

Research has indicated that most classroom norms are established in the first two months of the school year and remain relatively stable for the remainder of the year (McClain & Cobb, 2001; Wood, 1999; Wood, Cobb, & Yackel, 1991). Therefore, when identifying classroom norms, I used data from after the first two months of the school year, the time period when norms tend to have stabilized. I made a dozen observations of Ms. Skywalker’s classroom during this time period. However, during five of these observations, Ms. Skywalker’s class collaborated with another 5th grade class to work on mathematics.
Since the presence of another class changed the classroom community and social dynamics, I decided not to include this collaboration time in my data. Therefore, only parts of these five observations were included in my study. Thus, my entire data corpus for phase one consisted of seven full mathematics lessons and portions of five additional lessons. All observations were video recorded. This removed what Maxwell (1996) called the validity threat of description: relying on inaccurate or incomplete data. The video recordings gave me a complete record of all the whole-class dialogue on the days that I observed. The single video camera used suffices for my purposes since the only dialogue I focused on was whole-class dialogue. Recall that norms are constructs defined on a group level rather than an individual level. Although norms are characteristic of the group, they may not necessarily be characteristic of every individual within the group. It is certainly possible that Norm X may be characteristic of a class of 25 students while two students within the class do not hold Belief X. For this reason, I focused analysis only on whole-class discourse and not the interactions of smaller subgroups within the class.

Data analysis for phase one: identifying norms

In order to identify norms, one must understand the nature of them. What are they and how do they express themselves? My method for identifying norms stems directly from my definition of them given in chapter 1. At this point, it will be helpful to review some aspects of this definition since its implications here are significant. Norms are defined on a group level with the “group” viewed as a single entity. As in previous chapters, for linguistic clarity, beliefs will always correspond to the individual and norms will always correspond to the group. Social norms are constructs with both descriptive and evaluative components: they describe group behavioral regularities and also determine social expectations for acceptable group behavior. Sociomathematical norms are a subset of social norms that necessarily require specific mathematical content knowledge in order to be understood.\footnote{For a more detailed discussion of social and sociomathematical norms, the reader should refer to chapter 1.}

How were norm identified when analyzing video data? Since norms describe behavioral regularities in social interactions, I looked for exactly this in the video data. Several points require
elaboration however. First, when looking for regularities in group behavior, I focused my analysis on student actions. By “actions” in this case, I mean either discursive actions (i.e. speaking) or physical actions (e.g. writing something on the board). These student actions may be the explicit focus of the whole-class discourse, or they only be implicitly performed during whole-class discourse. Why focus on student actions and not on the teacher’s actions? Recall the discussion of norms from chapter 1. There is an inherent ambiguity when discussing “a group’s characteristics.” When we use the term “group”, we are generalizing about a collection of individuals. A group’s characteristics are comprised of the constituent individuals’ characteristics, viewed as a unity. When we say that a group “believes” Claim X, we are not necessarily saying that every constituent individual believes Claim X. Rather, we are implying that most group members, or “almost everybody” (Sfard, 2007, p. 539), believe Claim X. There may very well exist a small percentage of dissenters. This is the unavoidable ambiguity when generalizing about a collection of individuals. In any classroom community, and certainly in Ms. Skywalker’s class, the students constitute “most” of the community. In fact, they constitute every member of the community except one (the teacher), so it would be accurate to say that the students constitute “almost everybody” (Sfard, 2007, p. 539) in the community. Therefore, if there is evidence that the students, generally speaking, seem to demonstrate Norm X, then it is reasonable to conclude that Norm X is established within the classroom community, regardless of whether the teacher demonstrates it. For this reason, I argue that it sufficient to focus my analysis on student actions alone. Analyzing teacher actions during phase one could potentially be misleading. For example, assume that Ms. Skywalker’s actions repeatedly demonstrate Behavior Y. Assume also that, as is typical for teachers, Ms. Skywalker contributes a large percentage of the classroom discourse. Behavior Y would then appear prominent in the classroom discourse. However, this does not imply that any of the students are demonstrating Behavior Y. Indeed, none of them may demonstrate Behavior Y, meaning that Norm Y is certainly not established within the classroom community despite its apparent prominence. Focusing on student actions not only suffices then, but is more likely to result in an accurate identification of classroom norms. Hence, the focus on student actions helps to increase the validity of the phase one findings. This does not mean that the teacher’s actions are
ignored. Ms. Skywalker’s actions will receive primary focus in phase two of the study when I investigate teacher strategies used to establish norms.

In addition to focusing on student actions, I also focused on student actions that occurred specifically during whole-class mathematical discourse. Why whole-class mathematical discourse? This again is due to the inherent limitations in using group language. Group regularities are demonstrated by “almost everybody” (Sfard, 2007, p. 539) within the group. This means that the behavior of a small subgroup of the class (such as a pair of students) may not accurately reflect the norms of the entire class. To identify the behavioral regularities demonstrated by almost everybody, I must maintain my focus on the collective interaction of almost everybody. Why whole-class mathematical discourse? The first research question is specifically focused on norms associated with productive mathematical discourse. Norms associated with other subjects (e.g. spelling, social studies), while interesting, may not necessarily be associated with productive mathematical discourse.

**Noteworthy and ubiquitous norms**

Also when analyzing video data, I focused on student actions that evidenced *noteworthy* norms rather than *ubiquitous* norms. I define ubiquitous norms to be norms commonly found across the U.S. educational system that primarily support basic classroom routines and logistics. For example, during whole-class mathematical discourse, Ms. Skywalker’s students usually remain in their seats. This behavioral regularity evidences a norm that students should sit in their seats during class discussion (rather than sitting on their desks or the floor, or moving around the classroom). While factually accurate, this is not particularly enlightening or helpful. Thus, I would classify this norm as a ubiquitous norm since, in the context of the general U.S. educational system, it is not unusual. Other examples of ubiquitous norms would be students raising their hands when they wish to speak, not talking when the teacher is talking, obeying teacher commands, and other norms associated with the traditional I-R-E form of discourse. By contrast, a noteworthy norm is a norm that is unusual or uncommon when viewed within the context of the general U.S. educational system. The difference between a student action that supports
a ubiquitous norm and a student action that supports a noteworthy norm can be subtle. Consider the student actions\(^8\) in the following two examples:

Teacher: What was your answer for problem one?
Student 1: I got 42 miles.

Teacher: What was your answer for problem one and how did you get it?
Student 2: I got 42 miles. I multiplied 6 times 7.

At first glance, it may appear that the two student actions differ. Student 1 provides only an answer while Student 2 provides an answer and a supporting computational step. We may be tempted to conclude that the two examples evidence two different norms: the first a norm of providing only answers and the second a norm of providing supporting computations. However, in both examples, the students are doing exactly what the teacher has requested. For this reason, I argue that both examples support the same ubiquitous norm: students obey the teacher’s commands. Now consider the next two examples:

Teacher: What was your answer for problem one?
Student 3: I got 42 miles. I multiplied 6 times 7.

Teacher: What was your answer for problem one and how did you get it?
Student 4: I got 42 miles.

From a cursory glance, these examples appear quite similar to the previous two. I argue however that unlike the previous examples, these examples do not evidence ubiquitous norms. Notice that in both examples, the students did not do exactly what the teacher requested. In the first example, the teacher requested only an answer, but the student supplied a supporting computation as well. If a similar phenomenon is observed with other students in the classroom, then there is evidence that a norm of providing supporting computations exists. In asserting that “a norm exists”, I am asserting that a regularity in behavior and a corresponding expectation for this behavior exist on a group level. I am not

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8 Remember that I am using “action” in a broad sense here. It could be a physical action (such as getting up and writing something on the board), or a discursive action (saying something).
asserting anything about the attitudes or feelings of group members. Students who demonstrate the same behaviors and expectations might inwardly experience very different attitudes towards these things.\(^9\) Regardless, they still demonstrate the same norm. In the second example, the teacher specifically requests a supporting computation which the student fails to provide, evidencing a potential norm that supplying only answers is sufficient. To summarize, if a student does exactly what the teacher requests, then that student action evidences a ubiquitous norm. To evidence a noteworthy norm, a student action, or a particular quality of a student action, must be unelicited by the teacher. For example, consider the following interactions from Ms. Skywalker’s class.

```
Student Leader\(^{10}\): Okay, who has the answer for the second problem? Emmett?
  Emmett: I got 10 weeks and 4 days
  Student Leader: Does everybody agree?
  Emmett: Can I show my work?
  Student Leader: Sure.

Student Leader: Did anyone get anything different? Arnold?
  Arnold: Well, I got 26 and three-fifths, or 26.6. Can I show how I did that on the board?
  Student Leader: Sure
```

Note that in both of these examples, the students spontaneously request to show their work after providing their answers. This is evidence that a norm of showing work might exist in Ms. Skywalker’s class. Even if a student action has been elicited by the teacher, a particular quality of that student action might be unelicited. In the following example from Ms. Skywalker’s class, several students have just explained their work on a particular problem. Ms. Skywalker then asks Ralphie, “how about you?\(^1\)”, indicating that she wishes for Ralphie to share his thinking as well.

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Ms. Skywalker: How about you?
  Ralphie: Okay, so I did something pretty similar to Arnold… I got 16 times 8 is 128. I
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\(^9\) To investigate students’ attitudes and feelings about norms would require either surveys or numerous interviews. Regardless, investigating this particular area is outside of the scope of this study.

\(^{10}\) The student leader is responsible for leading the whole-class review of routine morning work. This is not always the same student. The responsibility of student leader rotates through the entire class, so that every student will be student leader multiple times throughout the school year.
don’t know my 16s [times tables] very well, so I just did the standard algorithm: 16 times 8 is 128.

Although Ralphie’s sharing is a teacher-elicited action, two particular qualities of his response were not specifically elicited. First, Ralphie compares his thinking to Arnold’s. Second, Ralphie’s response indicates metacognition: he explicitly states that used the standard multiplication algorithm because he does not know his times tables for 16 very well. Since Ms. Skywalker did not ask Ralphie to compare his solution to others or articulate metacognition, these qualities of Ralphie’s response evidence noteworthy norms: students should use articulate their metacognitive thoughts and students should compare their solution strategies with others. This example illustrates as well that one student action may evidence multiple norms simultaneously. Finally, noteworthy norms do not necessarily have to stem from student actions that are the explicit focus of discussion. Although this has been the case in all of the examples given thus far, it need not be. Implicitly performed student actions, if these actions seem to contribute to or enable discussion in a particular way, could reflect noteworthy norms. For example, the fact that students raise their hands when they wish to speak is ubiquitous, since this norm is found across the U.S. educational system. However, if students use some sort of hand signals to communicate additional information to the teacher when raising their hands (e.g. holding up one, two, or three fingers), then this would be a noteworthy norm even though it is not the explicit focus of discussion. This focus on unelicited student actions or unelicited qualities of student actions contributes towards the validity of my hypothesized norms by providing specific criteria when analyzing video. Rather than simply “analyzing” the video in a vague and unspecified sense, I will be looking for clearly defined events.

Open coding

Having now described the data corpus for phase one as well as the idea of unelicited student actions that evidence noteworthy norms, I now survey the overall process of data analysis. This process utilizes certain principles and methods from a grounded theory approach to research. Grounded theory seeks to generate new, generalizable theory based on (or grounded in) data that has been collected
In identifying norms that characterize Ms. Skywalker’s class, I am, in a sense, constructing theory: I am using the construct of norm to explain observed regularities in the class’s behavior. However, the “theory” that I am constructing is not applicable to all classrooms in general, but only to Ms. Skywalker’s class in particular. Therefore, while some principles from grounded theory were employed, others were not. The data analysis process consisted of repeated cycles of open coding followed by axial coding. During open coding, the video data was broken down analytically as unelicited student actions (or qualities of unelicited student actions) were conceptually labelled using the qualitative data analysis software, ATLAS.ti. These labels comprised the hypothesized norms. As in grounded theory, every hypothesized norm introduced during open coding was initially considered provisional until earning its permanence by repeated supporting evidence and a lack of significant disconfirming evidence (Corbin & Strauss, 1990). The constant comparative method was the process by which the hypothesized norms were continually compared against the video data. Since many studies have already investigated social and sociomathematical norms, many specific examples of them are given in the research literature.

Table 3-1 below provides a list of some of these previously-identified social and sociomathematical norms. The norms are not listed in any particular order, and the reader may note that some are quite general while others are more specific. This list served as a reference as I began the process of generating labels for the unelicited student actions in the data. This does not mean that I limited myself to using only these labels or that I expected to find all of these norms in Ms. Skywalker’s classroom. Rather, the list simply helped to facilitate the initial process of finding suitable labels for the behaviors that I witnessed.

<table>
<thead>
<tr>
<th>Justifying</th>
<th>Challenging claims</th>
<th>Metacognition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explaining thinking/reasoning</td>
<td>Reinterpreting a problem/solution</td>
<td>Connecting different mathematical ideas/concepts</td>
</tr>
<tr>
<td>Probing the thinking of others</td>
<td>Anticipating possible solutions</td>
<td>Contextualizing/decontextualizing</td>
</tr>
<tr>
<td>Explaining the rationale behind computations</td>
<td>Checking the reasonableness of a solution</td>
<td>Comparing different problem situations</td>
</tr>
</tbody>
</table>
Critiquing the thinking of others  Comparing different strategies  Expressing a disposition towards mathematics
Generalizing  Conjecturing  Providing counterexamples
Exploring incorrect reasoning in a mistake  Creating representations of problems  Using mathematical terminology or notation
Comparing representations of problems  Refining language/claims to be more precise

One point here requires further elaboration. As mentioned earlier, when performing open coding, I looked for either unelicited student actions or *qualities* of unelicited student actions. However, some unelicited qualities did not become apparent until later on in the analysis. Hence, working through the phase one data was not a strictly linear process, but was rather somewhat iterative. Already-analyzed data was revisited as necessary when new codes emerged midway through the data corpus.

**Disconfirming Evidence**

A crucial part of the constant comparative method is identifying potential disconfirming evidence. Disconfirming evidence is not the mere absence of confirming evidence, but rather is evidence that directly contradicts the proposed codes. What would disconfirming evidence for a hypothesized norm look like? Situations where a student violates a hypothesized norm provide *potential* disconfirming evidence. I use the word *potential* because the reactions of other students will determine whether or not the incident is actually disconfirming. Recall that norms have an evaluative component: they are the criteria group members use to determine acceptable social behavior. If a student violates a hypothesized norm and no other students in the group react or indicate that anything is amiss, then the incident provides disconfirming evidence. However, if the other students do react and indicate that their expectations have been violated, then the incident actually provides *confirming* evidence for the norm. As an example, assume that I hypothesize a norm that students should explain their work when providing an answer. Consider then the following episode from Ms. Skywalker’s class:

Student Leader:  Who thinks they know the first answer for number one? Janet?
Student Leader: Could you show how you got your answer?
Janet: On the board?
Student Leader: Yes.

In this incident, Janet provides only an answer without any supporting explanation, violating my hypothesized norm. The student leader then prompts Janet to offer an explanation for her answer. The student leader’s action evidences an expectation that answers should be accompanied by a supporting explanation. This provides confirming evidence for my hypothesized norm. However, if the student leader had not prompted Janet to show her work and no other students had commented either, then the incident would provide disconfirming evidence against the hypothesized norm. Situations like this one will contribute significantly towards strengthening the validity of my hypothesized norms. When a hypothesized norm is violated, evidence will emerge either for or against the norm. How much disconfirming evidence is necessary in order for a hypothesized norm to be modified? This depends. No individual always acts in perfect accord with their beliefs. Similarly, I believe it is reasonable to assume that no group always acts in perfect accord with its norms. Consequently, the existence of a disconfirming incident will not automatically invalidate a hypothesized norm. If there is a substantial amount of confirming evidence for a particular norm and only a single disconfirming episode, then I will probably not modify the hypothesized norm. However, if there is less confirming evidence, then a single disconfirming episode will exert greater influence in deciding whether to modify the hypothesized norm.

Axial coding

After open coding a video, the next step in data analysis is to perform axial coding. During axial coding, the norms are examined further to more precisely distinguish their boundaries and their relationships to each other (Corbin & Strauss, 1990). By “boundaries” I mean things such as the conditions under which the norm occurs, the frequency with which these conditions arise, the various ways in which the norm is actually expressed, and what distinguishes the norm from a similar one. This also includes determining the nature of the relationships between the norms. Which norms are more
abstract and which are more concrete? Are some norms a specific expression of a more general norm? Are two norms so similar that they can be merged? Is another norm so general that it needs subdivisions? One possible indicator of this is the quantity of student actions supporting each norm. A large number of supporting student actions could mean that the norm is prevalent in the classroom, but it could also mean that the norm is too abstract and needs decomposing. Likewise, a norm with a small number of supporting actions could mean that the norm is not prevalent in the classroom. However, it could also mean that the norm is too concrete and needs to be abstracted and merged with others. This process of axial coding resulted in a sort of norm hierarchy, represented by Figure 3-1 below. As more specific, concrete labels were grouped together under more abstract labels, the result was norms consisting of sub-norms. Each of these norms in turn supported one (or more) of the defining qualities of mathematically productive discourse. These results will be presented in detail in chapter 4. Figure 3-1 is provided simply to help illuminate the process of axial coding and how it shaped the nature of the results.

After completing axial coding, I then proceeded to open code a new video, followed by more axial coding. This cycle repeated itself until phase one data analysis was complete.

Figure 3-1. A representation of the norm hierarchy created by grouping more specific labels under more abstract labels.
Data saturation

The phase one data corpus consisted of one dozen videos of Ms. Skywalker’s class from after the first two months of the school year. I began analysis with the videos nearest the end of the school year, the period when norms were most firmly established. I then worked backwards through the school year, open coding a new video followed by axial coding. Several times as new codes emerged, I returned to already-analyzed videos from the end of the school year and re-analyzed them in light of the new codes. The majority of the codes emerged within the first few videos. By the time I analyzed the last few videos, no new codes were emerging, and the boundaries of the norms and their relationships to each other were clearly defined. In addition, the new unelicited student actions were fitting well into current norm categories. This does not mean that there were no anomalous student actions. There were, in fact, a handful of “anomalies,” unelicited student actions that simply did not fit into any of the norm categories. However, the number of these anomalies was small, especially when compared with the dozens or hundreds of student actions supporting each norm. Thus, I concluded that I had reached data saturation, that my phase one data corpus was sufficient to provide an accurate representation of the case study classroom. In other words, it is reasonable to conclude that given additional data of Ms. Skywalker’s classroom, no radically new norms would appear. It is true in a study such as this that regardless of the size of the data corpus, additional data would always strengthen the findings and increase the validity of the study. However, there is an inherent tension between this desire for more data and the logistical constraints in collecting it. More data would always be welcome, but more data is not always practical or realistic. The phase one data corpus consisted of approximately ten hours of the case study classroom spread across twelve different days of observation. As a point of comparison, McClain and Cobb’s (2001) longitudinal study of social and sociomathematical norms relied on video data from six different mathematics lessons. Thus, given the lack of new code generation near the end of my data corpus, as well as the comparison with McClain and Cobb’s (2001) study, I believe that my data corpus suffices to reveal regularities and expectations in classroom interactions.
Interviews and member checks

A crucial component of a valid qualitative study, such as a case study, is *data triangulation*. This means obtaining data from multiple sources and considering multiple perspectives on the data from individuals besides the researcher to provide corroborating evidence (Creswell, 2012). In my study, this triangulation came through member checks with both the teacher and the students. Member checks with the teacher came in the form of regular interviews. On twelve different occasions throughout the 2014–2015 school year, I interviewed Ms. Skywalker either directly before or after my observation of her class. These interviews were fairly unstructured. Ms. Skywalker was willing and eager to talk about her classroom, her students, and her teaching philosophy in detail. In nearly every interview, a simple, general question (e.g. “How do you think your class is doing mathematically right now?”) was sufficient to prompt her to talk at great length about her lesson, her goals for the lesson, the parts of the lesson that she was satisfied or unsatisfied with, her students’ development, and what she hoped to change or further develop in her students. In the process, Ms. Skywalker often spoke about the norms that she hoped to develop for her classroom and whether she thought they were successfully established. She did not actually use the term “norm,” but rather used equivalent language that indicated the same idea. For example, she might say something like, “Some of my students still are not explaining their work unless I prompt them to do so. I’m going to have to work with them on that.” This statement indicates that Ms. Skywalker hopes to establish a norm of justification and that she believes this norm is only partially established at the moment. These sorts of comments from Ms. Skywalker during interviews provided another perspective on the case study classroom. Comparing my interpretation of observed regularities with Ms. Skywalker’s interpretation of her own classroom strengthens the validity of my findings.

Beyond Ms. Skywalker, an additional perspective on the case study classroom was provided by member checks with the students. There were four different occasions throughout the 2014–2015 school year when Ms. Skywalker solicited her class for general input about their mathematical learning. On each of these occasions, she asked a broad, non-leading question and then allowed her class to contribute many
suggestions while providing minimal commentary or evaluation, even if student responses were vague or redundant. The broad questions she asked that initiated each of these occasions were:

- **What do you have to learn before you can really start getting into math reasoning, do you think?** What do we have to do first? What are some, like, foundation skills that you have to have? I wonder. What do you guys think? (Nov. 13th)
- **I want to know what you guys learned by being the teacher [to your partner], or what you thought of… Anyone want to share something that they noticed or they felt or they feel how this—how things went, whatever.** (Nov. 17th)
- **I want you to turn to someone next to you and in small groups I want you to talk about what you remember from [long division], what was your key learning from this? Was there an idea that went away with that you were like, “Ohhh!” Turn, talk go.** (Jan. 15th)
- **What’s the first step of learning?** (Mar. 5th)

Because of the open-ended nature of these questions and the fact that Ms. Skywalker did not try to guide students to a particular answer, these occasions offered a unique glimpse into what ideas, skills, and expectations the students perceived to be important for their learning and their classroom environment. Hence, these four occasions functioned as a member check with the students. Combined with the teacher interviews, this offered yet another perspective on the norms and expectations present in the classroom. Thus, the validity of the study was increased through triangulation with both teacher and student perspectives.

**Phase two of the project**

Completing phase one of the study resulted in a list of identified norms for Ms. Skywalker’s class (the *productive norms*), thereby addressing the first research question. The second phase of the study focuses on the second research question by identifying teacher strategies used to establish these productive norms.

**Data collection for phase two**

Data for phase two came from 13 observations performed during the first two months of the 2014–2015 school year. As in phase one, all observations occurred during mathematics since my research goals are to identify norms and strategies associated with mathematical discourse. Additionally, twelve
different interviews with Ms. Skywalker over the course of the school year served two key purposes. First, they functioned as a member check, allowing me to triangulate my observed teacher strategies. Second, the interviews provided insight into the intentions underlying teacher actions, thereby illuminating more subtle teacher strategies.

**Data analysis for phase two**

The process of analyzing phase two data bore many similarities to the phase one analysis of norms. Grounded theory principles were again utilized during this process. For both interview and observational data, I focused specifically on teacher actions\(^\text{11}\) that seemed to promote the productive norms. It is important to emphasize that my goal in phase two was not to exhaustively identify every strategy employed by Ms. Skywalker. Rather, I only sought to identify strategies intended to support the productive norms. As in phase one analysis, I focused on teacher actions that impacted the class as a whole. When dealing with individual students (not during whole-class discourse), Ms. Skywalker may have modified her strategies depending on her knowledge of that individual. My second research question does not focus on identifying customized strategies for each individual, but rather general strategies used when dealing with the class as a whole. The data analysis process consisted of repeated cycles of open coding followed by axial coding. During open coding, video data from observations and audio data from interviews were broken down analytically as teacher actions were labelled using the qualitative data analysis software, ATLAS.ti. These labels comprised the hypothesized strategies. Following grounded theory principles, every hypothesized strategy introduced during open coding was initially considered provisional. A hypothesized strategy had to earn its permanence, via the constant comparative method, through clear supporting evidence and a lack of significant disconfirming evidence. What would disconfirming evidence for a hypothesized strategy look like? The main source of disconfirming evidence

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\(^{11}\) I use the phrase “teacher actions” here in the same sense that I used “student actions” earlier in chapter 3. Teacher actions include physical actions (like writing on the board), discursive actions, and decisions made prior to class.
was interviews: if Ms. Skywalker told me that she did not use a strategy, then that was clear disconfirming evidence.

After open coding a video or interview, the next step in data analysis was axial coding. This involved clarifying details such as the following: Under what conditions is the strategy used? How does the strategy actually express itself? Are there trends over time in the use of strategies? What distinguishes a strategy from a similar one? Are some strategies more abstract while others are more concrete? Are some “smaller” strategies a component of a more encompassing one? Are two strategies so similar that they can be merged? Is another so general that it needs subdivisions? One possible indicator of this is the quantity of teacher actions supporting each strategy. A large number of supporting teacher actions could mean that the strategy is favored by Ms. Skywalker, but it could also mean that the strategy is too abstract and needs decomposing. Likewise, a strategy with a small number of supporting actions could indicate infrequent use. However, it could also mean that the strategy is too concrete and needs to be abstracted and merged with others.

After completing axial coding, I then proceeded to open code a new video or interview, followed by more axial coding. This cycle repeated itself throughout the data analysis process. As in phase one, this process of working through the data corpus was not strictly linear, but was somewhat iterative in nature. As new codes emerged, I returned to past data as necessary to re-analyze it in light of these new codes. Unlike in phase one however, I did not consider the possibility of data saturation. Previous research has identified this initial two month period of the school year as a time of flux when classroom norms are being established (McClain & Cobb, 2001; Wood, 1999; Wood, Cobb, & Yackel, 1991). Furthermore, identifying trends in teacher strategies over the course this two month period is part of my goal. For example, Ms. Skywalker may use certain strategies predominantly during the first month of the school year and then employ different strategies during the second month as norms are established. Identifying

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12 Recall that the purpose of axial coding is to further examine the strategies and flesh out their boundaries and relationships to each other.
trends (or the lack thereof) during the initial flux of the school year required that I analyze data across the entire two month period.

**Macro and micro strategies**

Varying degrees of context must be considered when interpreting teacher actions. Consider the following dialogue from Ms. Skywalker’s class:

Ms. Skywalker: Everyone should be on page 438. Where is it on the column on the left hand side? About halfway down? Where is it?

How should this teacher action be interpreted? What is Ms. Skywalker trying to do? If I consider the immediate context, it becomes apparent that the class is looking up the definitions of *prime* and *composite* in their math textbook. But *why* is this information being sought? If I consider additional context, it becomes apparent that the class is determining if the number 15 is prime or composite. The claim “15 is composite” is on the table and the class is developing a precise justification for it. But *why* the class is trying to determine the primality of 15? Considering even more context reveals that the class is investigating the number 15 and its various properties. From these examples, it should be evident that interpreting teacher actions depends on the amount of context considered. If a narrow amount of context is considered, then the class is looking up the definitions of mathematical terms. If a broader amount of context is considered, then the class is explicitly establishing a basis to use in justifying a claim. If an even broader amount of context is considered, then the class is investigating and justifying properties of the number 15. These different interpretations are nested within one another: The class is justifying properties of 15. To do this, they are establishing a basis to use in a justification. To establish this basis, they are defining relevant mathematical terms. To define these terms, they are looking up information in their math book. This example illustrates the need to consider various levels of context when interpreting teacher actions.
I divided teacher strategies into two general categories based on the amount of context considered. *Micro strategies* included teacher actions in light of their more immediate context. Often, this immediate context spanned several minutes, but never extended beyond that particular day. Examples of micro strategies include direct prompts and highlighting a positive example. In general, micro strategies tended to exert a direct and immediate effect in promoting the intended norm and were employed by Ms. Skywalker in light of the immediate circumstances of the moment. By contrast, *macro strategies* included teacher actions in light of a broader context that potentially spanned days or even weeks. An example of a macro strategy includes selecting a task that strategically builds on and extends ideas introduced in previous lessons. In general, macro strategies were chosen in light of the overall, long-term mathematical development of the class, rather than immediate circumstances. These strategies tended to have an indirect effect in promoting the norms and were used for their long-term effects. To clarify the distinction between macro and micro strategies, they may be considered as Ms. Skywalker’s strategies when viewed through a “wide-angle lens” and a “zoomed-in lens” respectively. Macro strategies correspond to the wide-angle lens because they are viewed in light of the class’s development over months. Micro strategies correspond to the zoomed-in lens because they are viewed in light of the immediate context. These different “lens sizes” mean that it is entirely possible that a particular teacher action may simultaneously support both a micro and a macro strategy. When considered within a narrow context the action may support a micro strategy, but when considered within a broader context, the same action may be part of a much larger progression of mathematical development and hence support a macro strategy.

**Validity**

Specific measures that strengthen the validity of my findings have been explained throughout the methodology chapter. These measures will be summarized here. First, the *quantity* of my data helps to support validity. Over the course of the 2014–2015 school year, I completed 25 different observations of Ms. Skywalker’s classroom for a total of approximately 30 hours of observed mathematical activity. This large amount of data helps ensure that I obtained an accurate sense of what is “normal” for Ms.
Skywalker’s classroom (Merriam, 1998). Spending a large amount of time in the classroom also minimizes the validity threat of reactivity by familiarizing the teacher and students with my presence. For phase one, data from the first two months of the school year will not be used, thereby avoiding a time period when norms tend to be unstable (McClain & Cobb, 2001; Wood, 1999; Wood, Cobb, & Yackel, 1991). A record of the number of supporting student actions for each norm and supporting teacher actions for each strategy was kept. This provides one way of summarizing the relative amount of supporting evidence for each norm or strategy. Interviews with Ms. Skywalker allowed me to triangulate my findings by comparing my interpretation of the data with her interpretation of the classroom and her own teaching strategies. Further triangulation of observational data was made possible through the four student member checks described earlier in the chapter.

**The methodology and the research questions**

I now summarize how the methodology helps answer the original research questions. Recall the original research questions:

1. **What social and sociomathematical norms are associated with mathematically productive discourse?**

2. **What strategies can a teacher use to establish these norms in their classroom?**

The first research question seeks to identify norms that are associated with mathematically productive discourse. Recall that social norms are defined on a group level. They are a construct with both descriptive and evaluative components: they describe regularities in social behavior and serve as the criteria the group uses to determine acceptable social behavior. Sociomathematical norms are a subset of social norms that necessarily invoke specific mathematical content knowledge. Norms are identified from observing regularities in group behavior. Determining “regular” group behavior requires many observations of the same group over time. This makes an in-depth, longitudinal case study of a particular classroom an appropriate choice. However, not just any classroom will suffice. To identify norms associated with mathematically productive discourse, my study requires a classroom where
mathematically productive discourse frequently occurs. I have evidence that my chosen classroom fulfills this requirement. Norms are demonstrated by almost everybody in the classroom community. Focusing on student actions ensures that I am focusing on the regularities exhibited by almost everybody in the classroom. Limiting my analysis to whole-class discourse ensures that I am analyzing group beliefs and not the beliefs of a subset within the group. Focusing on mathematical discourse ensures that I am identifying norms associated with mathematically productive discourse. The limitation of unelicited student actions, or unelicited qualities of student actions, eliminates actions that are performed merely to satisfy the teacher’s immediate prompting and helps reveal student-held expectations.

The second research question seeks to identify teacher strategies used to establish the norms from the first research question. Previous research has identified the first two months of the school year as the crucial time when most classroom norms are established (McClain & Cobb, 2001; Wood, 1999; Wood, Cobb, & Yackel, 1991). Therefore, an in-depth longitudinal observation of the first two months of the case study classroom is an appropriate choice. The teacher interviews and focus on the teacher’s actions ensure that analysis remains centered on teacher strategies. Previously identifying the productive norms in phase one further focuses my phase two analysis by allowing me to discern which teacher strategies are of interest and which are not.
Chapter 4: Phase One Results

Recall that this study was guided by two main research questions:

1. What social and sociomathematical norms are associated with mathematically productive discourse?

2. What strategies can a teacher use to establish these norms in their classroom?

This chapter addresses the first research question by describing the norms that were identified in the case study classroom. The next chapter will address the second research question by describing the teacher strategies that were used to establish the identified norms. This chapter and the next are both descriptive in nature. An in-depth interpretation of the results, their implications, and their relationship to the research literature will be considered in the final chapter.

Norms identified in the case study classroom

As was discussed in chapter 3, data for identifying classroom norms came from observations performed after the initial two months of the 2014–2015 school year. This is the time period when most classroom norms have already been established and will remain relatively stable (McClain & Cobb, 2001; Wood, 1999; Wood, Cobb, & Yackel, 1991). Also recall that observational evidence for classroom norms was based on unelicited student actions. These are student actions, either discursive (i.e. speaking) or physical (e.g. writing something on the board, making a hand signal), made during whole-class mathematical discourse, that were not specifically elicited by the teacher. The rationale for this was to identify behavioral regularities demonstrated by “almost everybody” (Sfard, 2007, p.539) in the classroom community. Since the students comprise almost everybody in the community, the behavioral regularities they collectively demonstrate are the actual norms of the classroom. Finally, I focused only on unelicited student actions that evidenced noteworthy norms. By “noteworthy,” I mean norms that are unusual or uncommon in the context of the general U.S. educational system. Students may regularly raise
their hands without being specifically prompted by the teacher, but this norm is found in nearly every classroom in the country.

As was also discussed in chapter 3, I interviewed Ms. Skywalker on twelve different occasions throughout the 2014–2015 school year to provide a means of triangulating my findings, thus increasing the validity of them. These interviews were relatively unstructured. Ms. Skywalker was eager to share about her perception of her own classroom and often only a single general interview question was sufficient to elicit a great deal of information from her. Additionally, a second and unexpected source of triangulation arose during the 2014–2015 school year. There were four different occasions—November 13th, November 17th, January 15th, and March 5th—when Ms. Skywalker solicited her class for general input about their mathematical learning. On each of these occasions, she asked a broad question and then allowed her class to contribute many suggestions while providing minimal commentary or evaluation, even if student responses were vague or redundant. The broad questions she asked that initiated each of these occasions were:

- What do you have to learn before you can really start getting into math reasoning, do you think? What do we have to do first? What are some, like, foundation skills that you have to have? I wonder. What do you guys think? (Nov. 13th)
- I want to know what you guys learned by being the teacher [to your partner], or what you thought of… Anyone want to share something that they noticed or they felt or they feel how this—how things went, whatever. (Nov. 17th)
- I want you to turn to someone next to you and in small groups I want you to talk about what you remember from [long division], what was your key learning from this? Was there an idea that went away with that you were like, “Ohhh!” Turn, talk go. (Jan. 15th)
- What’s the first step of learning? (Mar. 5th)

Because of the open-ended nature of these questions and the fact that Ms. Skywalker did not try to guide students to a particular answer, these occasions offered a unique glimpse into what ideas, skills, and expectations the students perceived to be important for their learning and their classroom environment. Hence, these four occasions functioned as a member check for the students. Students did not actually use words like coherency, justification, or active listening, but rather used age-appropriate equivalent terms such as “making connections”, “explaining your thinking”, and “following along.” These four occasions,
combined with the teacher interviews, provided a member check of both the teacher and the students, allowing me insight into their perceptions of the classroom.

In the upcoming sections, I will systematically introduce and explain the five norms that I identified in Ms. Skywalker’s classroom: *active listening, coherency, justification, computational strategies, and multiple perspectives*. Each of these five norms encompasses multiple sub-norms, which will also be explained. To better illustrate the sub-norms, I will provide both typical examples (*exemplars*) and exceptionally clear examples (*paragons*) as appropriate. Additionally, for each norm, I will present the member check data from both Ms. Skywalker and the students. After introducing each of the five norms and their constituent sub-norms, I will then present a visual model that represents the relationships between the five norms. This model, and the relationships between the norms, will be elaborated upon. Finally, I will discuss two “minor norms” I identified that did not demonstrate enough evidence to warrant equal consideration with the five “major norms.”

**Active listening**

*Students actively listen to others in the classroom. This is evidenced through the following sub-norms:*  
- Using hand signals to indicate agreement or disagreement, and convey other information  
- Sharing or revoicing a peer’s thinking  
- Noting similarities with previously shared ideas  
- Challenging others’ claims and assumptions

The norm of *active listening* was classified as a social norm since it does not necessarily require specific mathematical content knowledge in order to be understood. In fact, there is nothing intrinsically mathematical about this norm. Like its label implies, *active listening* means that students actively listen to others in the classroom. This is evidenced more specifically through four sub-norms. The most prominent sub-norm within *active listening* was students’ use of hand signals to indicate agreement or disagreement. Ms. Skywalker had taught her students the American Sign Language signs for “yes” and “no,” and students frequently utilized these to express either agreement or disagreement with an answer, strategy, question, or idea that had just been shared. Although “yes” and “no” were not the only signs Ms.
Skywalker taught her class, they were by far the most commonly utilized. Additionally, students sometimes used a two-thumbs-up, thumbs-up, or thumbs-sideways to indicate their level of understanding and learning.

Students most frequently shared or revoiced a peer’s thinking during a “pair-share” sequence. During the “pair” part of the sequence, students would talk with a partner about a concept, claim, or problem that had arisen in whole-class discourse. The time allotted for this typically ranged from five seconds to several minutes. Then in the “share” part of the sequence, Ms. Skywalker would call the students back to whole-class discourse and allow the various pairs to share their thoughts with the rest of the class. During this time, students would often share their partner’s thinking rather than their own. In the clearest examples they were explicit about the source of their ideas, such as in the following paragon where Tim’s explicit mention of Carlos leaves no doubt about the idea’s origin:

Ms. Skywalker: What do you notice about these two [numbers]?
Tim: Carlos told me that, um, that if you get a common multiple, then if you double it, you get a common multiple again.

In less-clear examples, students simply used “we,” implying that the ideas being shared were a combination of both individuals’ thoughts. In the following exemplar, Dorothy Ann uses “we” rather than “I” when sharing, intimating that her partners were involved in generating these ideas:

Ms. Skywalker: Dorothy Ann, Leah, Phoebe: share what you guys thought.
Dorothy Ann: So we came up with improper fractions and mixed numbers.

However, a reasonable objection could be raised that Dorothy Ann is simply sharing her own personal thinking while using “we” to create an appearance of collaboration. While this certainly could be the case, the member check evidence from both the teacher and the students suggests that it is not. In an interview later in the year, when discussing her overall impression of her class, Ms. Skywalker made the following remark:
Ms. Skywalker: If I call on someone, they don’t even say, “I think” anymore. They usually say, “We think”… And I very seldom— I don’t worry anymore when I say, “Turn and talk to your neighbor,” I don’t worry about what they’re talking about.

In this quote, Ms. Skywalker was alluding to the beginning of the year when students would often say, “I think…” when asked what the group had talked about. In her mind, the shift to, “We think…” indicated that students were listening to their peers. My own personal observations confirmed Ms. Skywalker’s assertion that, after the first two months of the school year, students did not say, “I think…” when asked to share their group’s thoughts. On another interview occasion, Ms. Skywalker also noted:

Ms. Skywalker: [The students] are used to listening to other people.

Thus, the member check with Ms. Skywalker supports the notion that the students’ use of “we” evidences active listening. Furthermore, the member checks with the students lend support to this as well. When asked about necessary foundational skills to engage in mathematical reasoning, students made the following two comments:

Janet: Being able to communicate your ideas with others and getting along with others.

Arnold: We’re all working together on things and so many people are talking about their own different strategies and using some strategies that are easier to explain [than mine].

Janet’s comment indicates that she views collaboration as an important foundational skill. Arnold states that his group experiences have been ones in which members are “working together on things” and in which his peers often employ more easily explainable strategies. Thus, both Janet’s and Arnold’s member checks agree with Ms. Skywalker’s perception that the use of “we” indicates active listening.

If relevant, students would note a similarity between their mathematical idea, strategy, or representation and one already shared. All instances of this sub-norm were fairly similar and were much
like the following exemplar in which Ralphie explicitly highlights the similarity of his computational strategy to Arnold’s:

Ms. Skywalker:  How about [your strategy for computing $3\frac{1}{2} \times 8$]?  
Ralphie:  Okay, so I did something pretty similar to Arnold… So first I converted three and one-fifths to an improper fraction which is $\frac{16}{5}$… I got 16 times 8 is 128… and I also knew 5 times 1 is 5.

However, Ralphie does not share how exactly his strategy was similar to Arnold’s, but only that it was. This was typical of other instances of this sub-norm. Without knowing the criteria students employed in judging “similarity,” it is impossible to determine whether they were noting superficial features (e.g. the overall appearance of a visual representation) or more noteworthy mathematical features (e.g. using an improper fraction rather than a decimal to perform a multiplication step). It must be emphasized as well that this sub-norm only encompasses similarities between strategies; there was no regularity of noting differences between strategies.

The final sub-norm under active listening was students challenging others’ claims and assumptions if they disagreed. The focus of these challenges ranged from challenging relatively simple mistakes to challenging tacit assumptions in claims. The following exemplar demonstrates a challenge of a “simple” mistake:

Ms. Skywalker:  [It’s] kind of like why 32 is odd. Remember how we talked about that with your parents: why is 32 odd? And they couldn’t really explain it.  
Janet:  32 isn’t odd.

In this case, Ms. Skywalker’s mistake is a simple slip of the tongue, accidently substituting the word “odd” in place of “even.” Janet quickly points this out and challenges the incorrect assertion. Many other challenges, however, were of a more complex nature. In the following exemplar, Carlos’s challenge of Ms. Skywalker’s claim demonstrates this complexity by focusing on an implicit assumption:

Ms. Skywalker:  Think about a test… You want to get 100% on your test. If you got 25% on your test how many questions did you get right? Stop, think… 100 questions on the test… You only got 25 right.
Carlos: For the test, you could be wrong because you might not get 25 questions right. Because what happens if there’s a question that’s worth 25 points?

In this instance, Carlos highlights an implicit assumption in Ms. Skywalker’s claim: All questions are equally weighted at one point apiece. Likewise, other challenges focused on tacit assumptions that claims were relying upon. Student member checks agreed with this sub-norm of challenging claims. When asked by Ms. Skywalker, “What’s the first step of learning?”, the following two comments were included among the responses:

Biff: Critiquing yours and other people’s answers.
George: Following along.

Biff’s comment indicates his view that answers need to be evaluated and, presumably, challenged if they are incorrect. George’s comment of “following along” does not directly mention a challenge or critique of claims, but it does indicate that George believes it is important to actively listen to others and understand their thought process.

Coherency

Students create mathematical coherency by identifying structural similarities and relationships across and within different math problems, situations, topics, operations, computations, notations, and visual representations. This is evidenced through the following sub-norms:

- Generalizing after observing patterns across multiple cases
- Identifying equivalencies across different notations, operations, units, and visual representations
- Recognizing when a particular problem manifests a more general mathematical property
- Transferring knowledge from a different problem
- Comparing numbers to landmark numbers to understand their relative size

Coherency was one of the four sociomathematical norms that I identified in Ms. Skywalker’s classroom. As its description above implies, students must have a certain amount of mathematical content knowledge in order to participate in this norm. Hence, coherency is a sociomathematical norm and not a social norm. In fact, coherency may be summarized by saying that students recognize structural
relationships between different mathematical objects. I use “objects” in a broad sense here, including such things as mathematical problems, topics, operations, computations, notations, numbers, specific cases, properties, and visual representations. *Coherency* is evidenced through five more specific sub-norms which will now be discussed in turn.

The most prominent sub-norm was student-voiced mathematical generalizing. Unlike most of the other sub-norms, the evidence for generalizing was not evenly distributed across different class sessions. Instead, it was primarily clustered within two different class sessions on November 13\(^{th}\) and November 20\(^{th}\). Within these two class sessions however, student-voiced generalizing was prevalent. In both of these class sessions, students were engaged in tasks that allowed for investigation of a phenomenon across many specific cases. On November 13\(^{th}\), the class was discussing the table shown below in Figure 4-1.

Each vertical and horizontal increment on this table resulted in the cell value increasing by 8 and 12 respectively. The class had just determined that moving to the left 2 cells and up 3 cells resulted in a cell of the same value and had specifically focused on cell values of 76 as an example. Arnold then spoke up and made the following generalization:

![Figure 4-1. Table discussed by the class on Nov. 13\(^{th}\).](image)
Arnold: If you make a chart of any numbers [as the horizontal and vertical increment values], let’s say [the vertical increment] is going up by 3 [for each row] and [the horizontal increment] is going up by 6 [for each column]

Ms. Skywalker: Listen! He’s trying to say if we make a chart of any numbers [for the horizontal and vertical increments].

Arnold: Then whatever common multiple of [the horizontal and vertical increments], let’s say, so [for] 3 and 6, a common multiple is 12. If you go [up] 4 [rows] with the [increment of] 3 [each time]… and then you go [left] 2 [rows with an increment of 6 each time], [the cell value] will stay the same because [the change in the cell value due to the vertical and horizontal shifts] is a common multiple [of both the horizontal and vertical increments], and 12 – 12 is zero.

Arnold’s generalization involves the structure of the table and finding cells with the same values. He recognizes that to find a cell of the same value, the value gained due to moving upwards on the table must be offset by the value lost from moving leftwards on the table; these equal but opposite values will be a common multiple of the horizontal and vertical increments. Arnold’s generalization is an exemplar of the student generalizations that were made. Although the content of Arnold’s generalization was more sophisticated than most of his peers’ generalizations, the fact that he spoke up and willingly shared a pattern was typical of Ms. Skywalker’s students. Also, the fact that Arnold’s generalization relied upon a visual representation was a common feature of this sub-norm as well. Other students extended previously-made claims by increasing the generality or domain over which the claim applies. In the following exemplar, Emmett, still referring to the table in Figure 4-1, extends the pattern of going up 3 cells and left 2 to find a cell of equal value:

Emmett: I want to prove Arnold’s claim even further.

Ms. Skywalker: Okay, give it a go. You want to stretch Arnold’s claim out a little bit? … Alright.

Emmett: Look at this. Arnold said that you go [left] 2 [columns] and up 3 [rows] and you get [the same cell value]. So [from (1,8)], I’m going to go even further… So I go over 2 [columns] and then I’m going to have –1 [for my x-coordinate]… and when I add 3 [vertical rows]… I get 11 [for my y-coordinate]. 11 [times 8 for each row] is 88… and then when you subtract 12 from 88, you get [the same cell value] again. So it went into the negative part [of the table], but [Arnold’s generalization] still works.
Emmett extended the domain of Arnold’s generalization into negative coordinates. This exemplar illustrates how student generalizations often built upon each other, creating a sequence of student-voiced generalizations. Occasionally when such a sequence of generalizations occurred, a “new” generalization was logically equivalent to, or even just a rewording of, a prior one. However, students did not usually seem to be aware of this. Student member checks corroborated the sub-norm of generalizations. In response to the general question, “What’s the first step of learning?”, two of the students responses included the following comments:

Marty:  Look for patterns.
Dorothy Ann:  On my math homework, my sister was helping me, and we were just looking for patterns… and it really does help to find the answers.

Since Ms. Skywalker’s students were 5th graders, they never used the word “generalization,” instead employing the age-appropriate term “patterns.” On many occasions, when making generalizations, students explicitly used the word “patterns.” Thus, it is reasonable to conclude that Marty’s and Dorothy Ann’s comments are referring to generalizing. These two comments then provide evidence that the students in Ms. Skywalker’s class viewed generalizing as a normative part of mathematical activity.

The second coherency sub-norm was identifying equivalencies between different mathematical “objects” such as notations, operations, units, and visual representations. Students recognized that the same quantities could be expressed via both fraction and decimal notation, and that the same ratios could be expressed via both fraction and percentage notation. They recognized equivalencies between operations, such as dividing by \( n \) and multiplying by \( 1/n \), subtracting \( x \) and adding \(-x\), and multiplying a number by \( y \) and repeatedly adding the number \( y \) times. The following two exemplars illustrate students identifying equivalencies between different mathematical objects:

Carlos:  Instead of 10 minus 2, it can be 10 plus negative 2.
Ms. Skywalker:  What?
Janet:  Um, a fraction is basically dividing the numerator by the denominator. So \( \frac{20}{7} \)
is the same as 24 divided by 3.

In the first exemplar, Carlos identifies an equivalency between operations: Subtracting a number is equivalent to adding its inverse. In the second exemplar, Janet identifies an equivalency between fractions and division: A fraction is equivalent to the numerator divided by the denominator. Other student actions supporting this sub-norm were similar; after the class had discussed a particular topic, a student would point out an equivalent way of representing a quantity, operation, or situation.

Students recognized when a particular problem manifested a larger, more general mathematical property, particularly when discussing computational strategies. For example, when discussing different ways to visualize the computation \(8 + (-5)\), students recognized a manifestation of the more general commutative property of addition:

Quinton: So I got \(-3\) because it’s \(5 + (-8)\).

Ms. Skywalker: Where’d you start on your number line?

Quinton: I started at \(-8\). And then I added up 5.

…

Ms. Skywalker: Keisha?

Keisha: I did the opposite of what Quinton did.

Ms. Skywalker: Show me what you did.

Keisha: So I started at positive 5.

Ms. Skywalker: Who started at positive 5? Put your fingers on positive 5. I don’t know how we’re going to end up at the same place if you guys are starting at different numbers. I don’t know how this is going to work. I don’t think it can.

Keisha, tell me what you did now.

Keisha: Since [Quinton] went in the positive direction, I’ll go in the negative direction.

…

Ms. Skywalker: Okay, I have to say one other thing. Arnold, say it.

Arnold: Um, it’s the commutative property of addition.

This paragon was exceptionally clear because after a detailed discussion of how \(8 + (-5)\) could be conceptualized multiple ways, Arnold neatly encapsulated the discussion by framing it as a specific instance of a more general property: the commutative property of addition. A more typical exemplar of this sub-norm is Quinton’s recognition of a structural similarity between unit conversion and place value:
Quinton: We thought that [converting days to weeks] was kind of like [converting between] tens, hundreds, and ones.

Although Quinton never explicitly articulates it, the larger mathematical property that he recognizes is grouping: A collection of smaller units (e.g. days, ones) can be grouped together to form one larger unit (e.g. weeks, tens). Quinton’s comment is a more typical exemplar of this sub-norm because although he recognized a manifestation of a more general property, he was not able to clearly articulate it.

Closely related to the previous sub-norm is the sub-norm of transferring knowledge from a previous problem to help solve a current one. With this sub-norm students recognize some sort of similarity inherent in both problems and strategically apply this knowledge to the problem at hand. In the following exemplar, the class had a worksheet with two similar problems on it. The first problem asked the students to calculate 25% of 36 dollars, while the second problem asked students to articulate the relationship between quarts and gallons. Jessie then spoke up, recognizing how one problem could be used to better understand the other:

Jessie: One quart is 1/4 of a gallon, and the equation that we’re supposed to write, 25% of 36, you could say 36 dollars is one gallon and the 25% is one quart.

Jessie’s statement indicates that she successfully transferred knowledge between the two problems rather than viewing them as unrelated entities. Student member checks supported this and the previous sub-norms. When asked, “What do you have to learn before you can really start getting into math reasoning?”, Ralphie replied with the following statement.

Ralphie: You could make connections to other problems, or you could make connections to other things

“Making connections” was the age-appropriate term the students used for establishing mathematical coherency. By “making connections to other problems,” Ralphie is corroborating the sub-norm of transferring knowledge from a previous problem to help solve a current one. His statement of “you could
make connections to other things” is more ambiguous. Ralphie could have in mind the sub-norm of equivalencies, or he could also be referring to the sub-norm of recognizing a particular manifestation of a more general mathematical property. Without further elaboration, it is impossible to assert exactly what Ralphie had in mind by “making connections to other things.” However, the statement does lend overall support to the idea of creating mathematical coherency.

The final *coherence* sub-norm was comparing numbers to “landmark numbers” to gain a sense of their relative size. Although Ms. Skywalker’s class never defined the term “landmark numbers,” their use of it indicates that they typically viewed landmark numbers as the nearest “round number” that they were more familiar with. Depending on the exact situation, this could vary from the nearest multiple of ten (when working with larger numbers) to the nearest whole number, nearest half, or nearest fourth (when working with fractions or decimals). The following exemplar demonstrates a straightforward use of a landmark number:

Ms. Skywalker:  About how big is 5/8? Leah?
Leah:  It’s a little bit more than 1/2 because maybe, because 4/8 is 1/2.

In this situation, Leah uses 1/2 as a landmark to gain a relative sense of the size of 5/8 through comparison. A more complex use of landmarks is given in the following exemplar, where Emmett uses 125 as a landmark to help calculate 128 ÷ 5:

Emmett:  The way I knew [the answer] was 25 3/5 without doing any multiplication was that I knew 5^3 is 125. And that’s 5 × 5 × 5. And 5 × 5 is 25. And I know that 25 × 5 is 125. So then I know that I have 25 with remainder 3/5.

In this exemplar, Emmett’s landmark was 125, which he knew was 5^3, or 25 × 5. Based on this landmark, he knew that 125/5 = 25. Consequently, he reasoned that 128/5 must equal 25 with a remainder of 3/5. The fact that Emmett mentioned “without doing any multiplication” confirms that he relied on making a comparison with the landmark rather than performing a mental computation. Further evidence for
landmark numbers arose during student member checks. When asked, “What do you have to learn before you can get into math reasoning?”, one response was the following:

Mary: Friendly numbers.

As with other terms, “friendly numbers” was the 5th grade equivalent for “landmark numbers.” When asked about their key learning from long division, one response was:

Jessie: You can multiply to narrow down the answer so that it’s way easier…

Jessie is referring to how multiplication can be performed prior to division to generate an upper and lower bound for the quotient. In effect, these two bounds function as landmark numbers that are used to evaluate the reasonableness of the answer. Thus, Mary’s and Jessie’s responses substantiate the sub-norm of landmark numbers.

Ms. Skywalker corroborated the overall coherency norm in my interviews with her:

Ms. Skywalker: Because of [my professional development], I look for [connections] that I hadn’t looked for before and, I know you know this about our class, we have this culture now that that’s just how we are.

Ms. Skywalker: [My students] can make the connections and they have the foundation skills and they’re going to, they’re building on things now.

Like her students, Ms. Skywalker used the term “connections” rather than “coherency.” Her statements show that she believed her students were creating mathematical coherency by recognizing similarities and relationships between different mathematical objects. Without further details from Ms. Skywalker, it is impossible to determine whether she had all five coherency sub-norms in mind when making these comments or only some of them. Regardless, the statements lend overall support to coherency.
Justification

*Students justify their claims, either verbally or on the board in front of the class. This is evidenced through the following sub-norms:*  
- Justifying by providing computational steps as well as a basis for those steps  
- Justifying through mathematical coherency  
- Justifying through visual representations  
- Justifying by contradiction

*Justification* was one of the norms that I identified in the case study classroom. Like *coherency*, *justification* is a sociomathematical norm. To participate in this norm, students must have a certain amount of mathematical content knowledge. To be coded as a justification, a student’s explanation had to be associated with a recognizable mathematical claim, either specific or general. It also had to fall within what Harel and Sowder (1998) call the analytical level, meaning that the justification appeals to a basis.\(^{13}\) Arguments that appealed to external sources of authority (e.g. teacher, textbook, respected peer) or to empirical reasoning (e.g. a collection of examples) were not considered justifications.\(^ {14}\) Four different categories of justification existed in Ms. Skywalker’s classroom. These sub-norms of justification will now be discussed.

The first *justification* sub-norm is justifying by providing computational steps as well as a basis for performing those steps. For a contextualized problem, this basis was typically the problem context itself. The following exemplar showcases this as Emmett provides his computational steps and basis for adding 5 weeks, 6 days to 4 weeks, 5 days:

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\(^{13}\) By basis, I mean a relationship, property, definition, strategy, visual representation, or feature of a problem context.  
\(^ {14}\) These sorts of arguments only occurred once in Ms. Skywalker’s classroom during all of my observations. She strongly emphasized from the beginning of the school year that an acceptable justification meant appealing to mathematical reasons.
Emmett: So when I did this first, I just added [the weeks and the days] up, straight up, and then I got 9 weeks and 11 days. And I know that there’s 7 days in a week, so I can make [the 9 weeks into] 10 weeks… So first I subtracted 7 days from 11 days and then I got 4 days. But, when I subtracted 7 days, I converted it into a week and then I added that into the [other] weeks. And then, I did 9 weeks plus 1 week, and 11 days minus 7 days, and then I got 10 weeks and 4 days.

Emmett’s justification consists primarily of listing his computational steps. However, his comment of “I know that there’s 7 days in a week” provides the basis for his computations. Emmett explicitly interprets what those computations are accomplishing within the problem context (“When I subtracted 7 days, I converted it into a week”). For decontextualized problems, the basis often consisted of relationships, properties, or facts that students appealed to. For example, in claiming that $9 \times \frac{3}{4}$ is equivalent to 6.75, Emmett provided his computational steps as well as a basis for them:

Emmett: I got 6.75 because first what I did was I divided 9 by 4 to find out what $\frac{1}{4}$ of 9 is. And then I multiplied that by 3 to figure out what $\frac{3}{4}$ of 9 is.

The basis in this situation is the relationship between fractions and the operation of division. Emmett recognizes and articulates that dividing by 4 results in $\frac{1}{4}$ of the original number. This sub-norm of justification by computational steps is similar to the norm of *computational strategies*. The justification sub-norm focuses on why students are providing computations (i.e. to justify a claim) while
computational strategies focuses on how students are performing the computations themselves (i.e. what strategies they are employing).

The second sub-norm was justifying through mathematical coherency. Recall that coherency included such sub-norms as recognizing equivalencies, recognizing when a more general mathematical property was manifested, transferring knowledge from a previous problem, and using landmark numbers. This sub-norm focuses on students using these coherency sub-norms for the purposes of justification. In the following exemplar, Rebecca uses the coherency sub-norm of equivalencies in justifying why \( \frac{5}{5} + \frac{5}{5} + \frac{5}{5} \) is not equal to \( \frac{15}{15} \).

Rebecca: \( \frac{5}{5} \) is also 1 whole, so 1 whole plus 1 whole plus 1 whole equals 3.

Here, Rebecca recognizes and utilizes the equivalency of \( \frac{5}{5} \) and 1 in her justification. Coherency would focus on the fact that this equivalency was recognized, while the justification sub-norm focuses on the use of this equivalency to justify a claim.

The third justification sub-norm was justifying through visual representations. The evidence for this sub-norm was not evenly distributed across the different class sessions, but rather was concentrated within a few sessions. This is likely due to the fact that certain tasks and questions were more conducive for visual representations while others were not. As an exemplar, Ms. Skywalker’s class investigated the differences of consecutive square numbers. The class soon determined that differences of consecutive squares resulted in consecutive odd numbers. To justify this claim students drew visual representations and appealed to them. Below, Figure 4-3 shows Liz’s representations of \( 5^2 - 4^2 \) and \( 6^2 - 5^2 \). For clarity, these representations have been recreated in Figure 4-4. The dark circles in Figure 4-5 represent the smaller square number, and the lighter circles represent the difference between it and next larger square number. Liz pointed out that the difference between squares will always form an L-shaped section, but that this section grows by two circles every time. She highlighted the two gray circles in Figure 4-4 as representing this increase over the previous difference.
The final justification sub-norm was justification by contradiction. Like the previous sub-norm, the evidence for this sub-norm was not evenly distributed, but was mostly concentrated within a single task. In this task, the class was trying to determine, given the limited information in Figure 4–5 below, how many crayons each pack contained. These constraints created a finite number of possible answers, as the class realized that the number of crayons per pack must be a factor of 36. This allowed the students to test a possibility and look for a contradiction, which is exactly what Janet did in the exemplar below:

Ms. Skywalker: She’s saying it can’t be 3 [crayons per pack]. Why Janet? Why can’t it be 3 and 12? Why can’t there be 3 in our packs?
Janet: Because then the 4 would be by the 12.
Figure 4-5. The table listing numbers of packs and total number of crayons that the class was using.

Janet had tested the possibility of 3 crayons per pack, knowing that 3 was a factor of 36 and therefore a potential solution. However, she quickly recognized a contradiction: the “4” in the left column was not next to the “12” in the right column, implying that 3 crayons per pack was not the case. Other justifications by contradiction during this task followed a similar process.

Member check data with both Ms. Skywalker and her students gave strong evidence that the norm of justification existed in the classroom. When asked about the first steps for learning and the necessary foundational skills, the following four comments were made by students:

Rebecca: Being able to explain your answer.

Dorothy Ann: You need to know how to be able to explain your answer.

Leah: Once [you’ve] figured out the answer, [you] have to know to, after the answer, to explain your answer.

Arnold: You have to be to explain your thinking to others, not just yourself.

Multiple students indicated that justifying, expressed in the age-appropriate phrase “explaining your answer,” was an expected part of mathematics in Ms. Skywalker’s classroom. Ms. Skywalker herself, after about two months into the school year, believed that the expectation of justifying was established:
Ms. Skywalker: I rarely have to tell [my students], “Don’t just come up with the right answer or what you think it is, but why or how? Explain.”

At the halfway point of the school year, she again expressed satisfaction with the expectations she had established in her classroom. However, at this point, she noted that a few students seemed to have either missed or forgotten her expectations for justifying:

Ms. Skywalker: I think [the expectations] are pretty much there [where I want them]… I still have to ask how and why sometimes to certain kids for reasoning and prompting.

Thus, considering the student and teacher member checks, it is reasonable to conclude that justification was upheld as a norm by almost everybody in the classroom.

Computational strategies

Students explicitly share the strategies they used to perform multi-digit computations. This is evidenced through the following sub-norms.

- Decomposing numbers by place value to perform computations with them
- Doubling numbers as part of a multiplicative computation
- Using landmark numbers to guide computations and check the reasonableness of them
- Using the inverse operation to perform a computation
- Transferring knowledge from a previous computation to perform a new computation
- Using equivalencies to change the computation into a more comfortable form
- Using the standard algorithms for various operations

Computational strategies was another sociomathematical norm in Ms. Skywalker’s class. This norm does not mean that students shared what computations they did or why they did them, but rather specifically how they did them. Therefore, the following example would not be an illustration of computational strategies:

Janet: Jessie and I were talking and she wondered, “I wonder how many hours [10 weeks and 4 days] would be.” So I converted it to days, which is $10 \times 7$ is 70, plus 4 days… and then I multiplied that by 24 because there are 24 hours in a day… and I found out… the answer is 1776 [hours].
In this example, Janet shares what computations she did (10 × 7, 70 + 4, and 74 × 24) as well as why she did them (to convert weeks to days, and to convert days to hours), but does not share how she actually performed any of them. If Janet had mentioned that she used the standard multiplication algorithm or that she had used a landmark number, then this example would support computational strategies. As an additional note of clarification, computational strategies applies only to multi-digit computations. For simple single-digit computations such as 3 + 6, or 5 × 4, or 8 ÷ 2, students did not usually provide details about how they performed these computations. Furthermore, this norm states that students explicitly share these computational strategies. Students may have been utilizing these strategies more frequently on their own, but computational strategies focuses only on when students shared their computational strategies in whole-class discourse. Seven sub-norms were identified, which will now be discussed.

The first sub-norm was decomposing numbers by place value in order to perform computations. This strategy was used primarily for multiplication or division, such as in the following exemplar where Janet shares how she uses decomposition to perform 52 ÷ 2:

Janet: I wanted to share my strategy for halving [the number 52].
Ms. Skywalker: Let’s hear your strategy for halving.
Janet: So what I do is, because 52 I don’t really know 52 in half. So what I do is: I halve 50 which is 25, and then I halve 2 which is 1. And then I do 25 + 1.
Ms. Skywalker: Janet: I actually kind of like that strategy.
Ms. Skywalker: Janet: And like 26, I do 10 and 3.

Although Janet does not actually use the word “decomposition,” her strategy relies upon it. She decomposes 52 into 50 + 2 and then divides each part separately to obtain 25 + 1. Recombining the two parts, she arrives at her answer of 26. Janet’s implicit use of decomposition was typical of most instances of this sub-norm. Students usually assumed the decomposition process, rather than explicitly mentioning it. Composition and decomposition of numbers was mentioned several times during student member checks. In response to the question, “What do you have to learn before you can start getting into math reasoning?”, student replies included the follow two comments:
Matt: Decomposing and recomposing.

Jennifer: The value of numbers.
Ms. Skywalker: Okay, the value of numbers. I kind of like that. What do you mean by “the value of numbers?”
Jennifer: Like, the value of 60 is 6 tens.
Ms. Skywalker: So a little bit of base-ten, place value?

These replies lend support to the existence of the sub-norm of decomposing numbers by place value.

For the second sub-norm, when performing multiplication, students seemed to prefer to use doubling when feasible. A paragon of this is Arnold’s use of repeated doubling to multiply by 8:

Arnold: And then I know $16 \times 8$ is 128. And I figured out that out because I know that $16 \times 2$ is 32. And then $16 \times 4$, or $32 \times 2$ is 64.
Ms. Skywalker: So he doubled it. He’s a good math kid, you guys. He uses doubling and halving all the time.
Arnold: And I doubled it another time, which 64 doubled is 128. Or doubling it 3 times is the same as multiplying it by 8.

Arnold was a particularly articulate student in Ms. Skywalker’s class. In this paragon, he clearly explains how doubling three times is equivalent to multiplying by 8. A more typical example of the use of doubling is the following exemplar. The class was investigating the ratio table shown in Figure 4-6, and Ralphie explained how he used doubling to determine that the value of the empty cell on the table was 80:

![Figure 4-6](ratio_table.png)

Figure 4-6. The ratio table listing number of cars and total number of tires that the class was discussing.
Ralphie:  There’s 40 and 10 [in row 2], so it’s doubling, and 10 got doubled.

To determine the empty cell, Ralphie compares row 4 of the table to row 2 and recognizes a multiplicative relationship of doubling. From this, he deduces that the empty cell must be 40 doubled, or 80. It is noteworthy that Ralphie did not compare row 4 to either rows 1 or 3, but specifically looked for a doubling relationship. Member checks support this preference for doubling students seemed to have when using multiplication. In response to the question, “What do you have to learn before you can start getting into math reasoning?”, the very first student reply was:

   Janet:  Doubling and halving.

In interviews, Ms. Skywalker confirmed that doubling was a common strategy that the class had used.

   Ms. Skywalker:  We used to have [computational] strategies listed on the board: Did you find a double? … You’ve got to use all those strategies…

These member checks corroborate the observational evidence that students seemed to rely on doubling when feasible for multiplicative operations.

The third computational strategies sub-norm is use of landmark numbers to guide computations or “narrow down” the range of possible answers. As explained earlier, landmark numbers were the nearest “round number” that students were familiar with. Depending on the exact situation, this could vary from the nearest multiple of ten (when working with larger numbers) to the nearest whole number, nearest half, or nearest fourth (when working with fractions or decimals). Dorothy Ann’s explanation of how she performed $9 \times 1/4$ is an exemplar of this sub-norm that relies upon repeated use of landmarks:

   Dorothy Ann:  I got 2.25 because I knew that 3 is um, is one-third of 9. And so then I thought maybe 2.5 would be it, but then [quadrupling it] I got 10… And then I tried, [4 times] 25 cents equals a dollar, so 4 of [those] would equal one more [onto your answer]. And then $2 + 2 + 2 + 2$ equals 8, and one more equals 9.
In computing $9 \times 1/4$, Dorothy Ann began with the fact that $9 \times \frac{1}{3} = 3$. Since $1/3$ is larger than $1/4$, Dorothy Ann knew her answer would be less than 3. Hence, 3 was the first landmark Dorothy Ann used in narrowing down the range of possible answers. She then tried 2.5, most likely because she viewed it as the next round number. However, multiplying by 4 revealed that 2.5 was still too large. Hence, 2.5 became the next landmark. From here, it is not clear if Dorothy Ann realized the answer had to be 2.25 or if she simply tried 2.25 as the next landmark and found that it worked. Dorothy Ann’s use of landmarks to “narrow in” on the answer is representative of how other students typically used them. In fact, this “narrowing” process was explicitly mentioned by one of the students during a student member check:

Jessie: [When doing long division], you can multiply to narrow down the answer so that it’s way easier…

Jessie is referring to how multiplication can be performed before division to generate an upper and lower bound for the quotient. In effect, these two bounds then function as landmark numbers that are used to “narrow down” the range of possible answers. When asked, “What do you have to learn before you can get into math reasoning?”, one student reply with the following:

Mary: Friendly numbers.

As with other terms, “friendly numbers” was the 5th grade equivalent of “landmark numbers.” Mary’s and Jessie’s responses substantiate the sub-norm of using landmarks as a computational strategy. Recall that landmark numbers were also a coherency sub-norm. In that case, the focus was on the fact that students identified landmarks and used them as a comparative reference point. For computational strategies, the focus is on the use of landmarks for the larger purpose of computation.

The fourth sub-norm was use of inverse operations to perform computations, primarily addition to aid with subtraction and multiplication to aid with division. In the following exemplar, Arnold explains how he used multiplication to perform $56 \div 4$: 
Arnold: I did two separate arrays.
Ms. Skywalker: Okay.
Arnold: One was... I know that 4 × 10 is 40 and that’s less than 56. So I did one array of 4 by 10. And then I see how much more do I need: 16. And then I know I’m doing rows of 4, so that’s a 4 by 4 because 4 × 4 is 16.

Rather than using a division algorithm as the framing of the problem might suggest, Arnold uses the inverse operation of multiplication and determines what number multiplied by 4 will yield 56. This exemplar is typical of the sub-norm in several ways. First, Arnold doesn’t explicitly mention the word “inverse” or justify his use of the inverse operation; he simply uses the inverse operation without any ado. Secondly, Arnold combines his use of the inverse operation with landmarks to “narrow in” on the answer. This use of inverse operations was also mentioned during a student member check:

Jessie: [When doing long division], you can multiply to narrow down the answer so that it’s way easier...

Although this comment was mentioned earlier in support of landmarks, it also supports the use of inverses because Jessie explicitly mentions the usefulness of multiplication in simplifying the work of division.

The fifth computational strategies sub-norm was transferring knowledge from a previous computation to aid a current computation. This sub-norm would only occur if a computation was structurally similar in some way to a recently performed one. Transferring knowledge from a previous problem was also a coherency sub-norm. In that case however, the focus was on the fact that students recognized some sort of feature inherent in both problems. For computational strategies, the focus is on students applying this knowledge specifically as a computational aid. In the following exemplar, the class has just finished solving 56 ÷ 8 and Liz had made a 8 by 7 array to assist her. Now the class is asked to compute 56 ÷ 4. Liz used her previous array, recreated in Figure 4-7 below:
Ms. Skywalker: Liz, tell us what you did here.

Liz: So I made the [8 by 7] array that I had before. And I knew that since 4 was half of 8, I could take off 4 [rows]—so the blue box right here—and then I just moved it up there because there’s also 4 [rows] there. And then I added 7 [columns] plus 7 [columns] and I got 14. And then I checked my answer: $4 \times 14$, and I got 56.

Liz recognized a similarity between the two problems: the second problem halved one factor and doubled the other compared to the first problem. Rather than recreating an entirely new array, Liz modified her first array, thus indicating that she had transferred knowledge from the first problem to the second to help her perform $56 \div 4$. Student member checks supported this strategy of transferring knowledge. When asked, “What do you have to learn before you can really start getting into math reasoning?”, Ralphie replied:

Ralphie: You could make connections to other problems, or you could make connections to other things

By “making connections to other problems,” Ralphie is corroborating the sub-norm of transferring knowledge from a previous problem to help solve a current one.

The sixth sub-norm is use of equivalencies to change a computation into a more comfortable form. Similar to the previous sub-norm, equivalencies was also a coherency sub-norm. However, the focus here is not on students merely recognizing equivalencies but using them as a computational aid. Janet does exactly this in the following exemplar when computing $3\frac{1}{5} \times 8$: 
Ms. Skywalker: Janet, will you explain your thinking? How you started this problem?
Janet: Sure. I don’t know how to multiply fractions, but I do know how to multiply decimals.
Ms. Skywalker: What do we got?
Janet: And so I converted $\frac{1}{5}$ into $\frac{2}{10}$ or 3.2.

Janet recognizes the equivalency between fraction and decimal notation and uses the more-familiar decimal notation to assist her with the computation. Similar to Janet’s example, students often converted between different equivalent notations (e.g. fractions, decimals, percentages, dividing by n, multiplying by 1/n, etc.) either to simplify computations or to check the reasonableness of computations. Ralphie’s member check comment, “you could make connections to other things,” lends possible support for this computational strategy. Certainly, when students convert notation, they are making a connection to “another thing.” However, because of the ambiguity of Ralphie’s comment, it is impossible to determine if this is exactly what he had in mind.

Finally, students also used the standard algorithms for addition, subtraction, multiplication, and division. There was no expressed stigma against doing this in Ms. Skywalker’s class, although students seemed to view the standard algorithms as fallback options when other computational strategies were unavailable. Ralphie’s comment when computing $16 \times 8$ reflects this ethos:

Ralphie: I got $16 \times 8$ is 128. I don’t know my 16’s very well, so I just did the standard algorithm: $16 \times 8$ is 128.

Ralphie’s comment about not knowing his 16’s and that he “just” did the standard algorithm seems to convey that it was not his initial strategy and was only employed due to impracticality of other computational options. The standard algorithms were not mentioned in any of the member checks with students or with Ms. Skywalker. This further supports the idea that they did not receive as much emphasis as other computational strategies.
Multiple perspectives

Students share multiple perspectives on how to conceptualize mathematical topics, relationships, and problems. Even if a correct solution strategy or visual representation has already been shared, students will volunteer new and different ways of solving it. This is evidenced through the following sub-norms.

- Sharing equivalent notation for expressing a quantity or concept
- Sharing multiple computational strategies
- Sharing multiple justifications
- Sharing multiple visual representations

*Multiple perspectives* was the fifth sociomathematical norm identified in Ms. Skywalker’s classroom. This norm means that students offered multiple ways to consider the various mathematical problems, topics, and ideas that arose in class. On some occasions, as the first sub-norm indicates, this meant sharing an equivalent notation for representing a specific quantity. A simple exemplar of this is Arnold’s answer to a straightforward multiplication question:

Arnold: Well, I got $26\frac{2}{5}$ or 26.6.

Arnold shares two different ways of representing the same quantity: one uses fractions while the other uses decimals.

The remaining sub-norms for *multiple perspectives* all tended to follow a similar pattern. They usually occurred when the class was discussing a math problem and one student had already shared a solution, representation, or justification. Other students would then continue to volunteer new ideas, methods, and representations. The remaining sub-norms all focus on the fact that multiple perspectives of something (e.g. computational strategy, justification, and visual representation) were being shared. For example in the following episode, the class had just finished computing the perimeter of a regular octagon with side length 5 by performing $5 \times 8$. Ms. Skywalker then asked what the perimeter of the octagon would be if the side length was increased to 6. Several students indicated that computing $6 \times 8$ would yield the perimeter of this new octagon. Jessie then spoke up and offered a different perspective on how to compute the perimeter of the larger octagon:
Jessie: You could’ve done $5 \times 8$ and then you added 8.
Ms. Skywalker: Why can you do that?
Jessie: Because $5 \times 8$ is 40, but then $6 \times 8$, you’re just adding another 8.

In this exemplar, Jessie offered a different perspective on a computational strategy. Member checks with both the students and the teacher supported multiple perspectives. When asked, “What do you have to learn before you can really start getting into math reasoning?”, two of the student responses were:

Emmett: Look at things from multiple perspectives.
Arnold: We’re all working together on things and so many people are talking about their own different strategies and using some strategies that are easier to explain.

Emmett’s response is a straightforward endorsement of multiple perspectives. Arnold’s response highlights one of the benefits he sees of multiple perspectives: some strategies are easier to explain than his. Both comments evidence the existence of multiple perspectives as a regular and expected part of mathematical activity in the mind of the students. Ms. Skywalker firmly believed that her class had embraced multiple perspectives. In an interview near the end of the year, she observed the following:

Ms. Skywalker: Now I can walk around and say, “I want to see a number line, I want to see a table, I want to see a whatever,” and look what I get.
Peter: You get all kinds of stuff.
Ms. Skywalker: I didn’t get that at the beginning of the year, did I? [The students] have learned through seeing other people and seeing other things to try multiple representations. Remember before, I said, “I want to see 2 representations.” At the beginning of the year, they were like, “What do you mean? That’s the way I did it.” “Well, show me a different way.” “Umm, there’s only one way, it’s the way I did it. I did it the right way.” You know that! Those were seriously the conversations I was having with kids. Now I say, “I want to see at least 2 representations,” they all go (motion of holding up papers for teacher to see). Because they know! But it has not always been that way.

Ms. Skywalker’s comments show that she is convinced her students have come to embrace multiple perspectives as a normative way of doing mathematics.
A model to represent the identified norms

Figure 4-8. A visual representation of the five norms identified in Ms. Skywalker’s classroom.

I put forth the model, Figure 4-8, as a way to visually represent the norms and the relationships between them. The box in this figure represents Ms. Skywalker’s classroom during mathematics. Inside are all of the noteworthy norms present when the class does math. The triple Venn diagram represents the noteworthy sociomathematical norms and the often-interconnected relationships between them. For explanatory purposes only, the regions within the Venn diagram are numbered. Some student actions only supported one of the sociomathematical norms (Regions 1, 2, and 3), while other actions supported multiple sociomathematical norms simultaneously (Regions 4, 5, 6, and 7). I will proceed to first discuss active listening, followed by each of the regions within the Venn diagram.

Active listening is positioned within the box but outside of the Venn diagram, indicating its status as a social norm. The fact that active listening surrounds the Venn diagram represents two things: first, that it is a “broader” social norm. It occurred during mathematics, but was not necessarily limited to
mathematical activity. Second, it represents the fact that *active listening* supported and undergirded the sociomathematical norms. Arnold’s response to Rebecca’s claim given below, exemplifies the supportive nature of *active listening* in relation to the sociomathematical norms. Rebecca had asserted that 10 weeks and 4 days is equivalent to 2 months, 2 weeks, and 4 days. Arnold then spoke up, challenging this claim:

Arnold: I was thinking about Rebecca’s months thing and I don’t think that it works because there’s only one month that has 28 days… and you’ve got two months, so there can’t be two months in a row with 28 days.

In this instance, Arnold demonstrates the *justification* sub-norm of justifying through contradiction. However, his entire justification occurs in response to Rebecca’s claim. He listened carefully to Rebecca, recognized an implicit assumption in her claim that a month has exactly 4 weeks or 28 days, and challenged it. In challenging, he put forth his own claim that Rebecca’s answer was invalid and then justified it by pointing out that only one month has exactly 28 days. If Arnold had not carefully listened to and understood Rebecca’s claim, he would never have offered his justification. Hence in this instance, *active listening* subtly enabled *justification*. In a similar manner, *active listening* led to students extending their peers’ generalizations, justifying as they responded to various claims, making connections between different peers’ work, and offering differing perspectives on the same mathematical topics. Although it was almost always in a subtle manner, *active listening* supported the other four sociomathematical norms identified in Ms. Skywalker’s classroom. Ms. Skywalker also saw *active listening* as building a foundation for mathematical reasoning. In the following interview excerpt, she was reflecting on the third Common Core Math Practice ("Construct viable arguments and critique the reasoning of others"): 

Ms. Skywalker: Critique the reasoning of others, I think I’m stepping there. Listening to others is the first step. Now comparing your thinking to someone else’s… that might be this next step.

Peter: Yeah, and they have to get their reasoning out there before they can start critiquing it.

Ms. Skywalker: Exactly! So we have first steps first.
In labeling listening as a “first step,” Ms. Skywalker implies that it plays a supporting role in enabling mathematical practices, such as critiquing others’ reasoning, to successfully emerge. Thus, it appears that active listening undergirded the sociomathematical norms within Ms. Skywalker’s classroom.

Region 1 in the Venn diagram represents student actions which support coherency and none of the other sociomathematical norms. An exemplar is the following student comment:

Quinton: We thought that [converting days to weeks] was kind of like [converting between] tens, hundreds, and ones.

Here, Quinton recognizes an underlying structural similarity between unit conversion and place value; both entail the idea of grouping smaller units into larger units. Quinton’s comment would therefore support the coherency sub-norm of recognizing when specific problems manifest more general mathematical properties. However, Quinton does not make this comment as part of a justification or computational explanation, nor is he trying to offer a different perspective on unit conversation. Rather, the goal of Quinton’s comment is simply to point out a structural similarity between two mathematical objects. Therefore, Quinton’s comment supports coherency, but not justification, computational strategies, or multiple perspectives. In general, student actions that fell into Region 1 in Figure 4-8 were actions that evidenced coherency for the sake of coherency; that is, these student actions were not embedded within a justification or a computational explanation, nor did they seek to offer a different perspective on something.

Region 4 on the Venn diagram represents coherency that is established within the context of a justification. In other words, these are student actions that support coherency for the larger purpose of supporting a justification. The entire justification sub-norm of justifying through mathematical coherency resides in this region. For more details, the reader should refer to earlier in the chapter when this sub-norm was discussed.

Region 5 on the Venn diagram represents connections that are made within the context of performing a computation. In other words, these are student actions that support coherency for the larger
purpose of supporting a computational strategy. The following *computational strategy* sub-norms reside within this region:

- Using landmark numbers to guide computations and check the reasonableness of them
- Using the inverse operation to perform a computation
- Transferring knowledge from a previous computation to perform a new computation
- Using equivalencies to change the computation into a more comfortable form

For examples of how *coherency* supported *computational strategies*, the reader should refer to the earlier discussion of these sub-norms. In a few rare cases however, this relationship was reversed: *computational strategies* supported the creation of coherency. This is precisely what happens in the following example.

The class’s discussion of computational strategies for $5 + (-8)$ leads to creation of mathematical coherency:

Quinton: So I got $-3$ because it’s $5 + (-8)$.
Ms. Skywalker: Where’d you start on your number line?
Quinton: I started at $-8$. And then I added up 5.

Ms. Skywalker: Keisha?
Keisha: I did the opposite of what Quinton did.
Ms. Skywalker: Show me what you did.
Keisha: So I started at positive 5.
Ms. Skywalker: Who started at positive 5? Put your fingers on positive 5. I don’t know how we’re going to end up at the same place if you guys are starting at different numbers. I don’t know how this is going to work. I don’t think it can. Keisha, tell me what you did now.
Keisha: Since [Quinton] went in the positive direction, I’ll go in the negative direction.

Ms. Skywalker: Okay, I have to say one other thing. Arnold, say it.
Arnold: Um, it’s the commutative property of addition.

The discussion of different computational strategies for $5 + (-8)$ allowed Arnold to create mathematical coherency by recognizing a manifestation of the commutative property of addition. Hence, *computational strategies* supported *coherency*. However, it must be stated that only a few actions in Region 5 followed this pattern. For the vast majority of actions in Region 5, *coherency* helped to support *computational strategies*. 
Region 2 in the Venn diagram represents student actions that supported *computational strategies* but none of the other sociomathematical norms. The actions in Region 2 ended up consisting of explanations of decontextualized computational problems. In these actions, students explained *how* they performed the relevant computation, but did not utilize mathematical coherency to do so. Furthermore, no mathematical basis for performing the computation or a sub-component of the computation was given. Hence, these explanations ended up supporting only *computational strategies*. An exemplar of this type of action is Ralphie’s explanation of how he computed $16 \times 8$:

Ralphie: I got $16 \times 8$ is 128. I don’t know my 16’s very well, so I just did the standard algorithm: $16 \times 8$ is 128.

Ralphie explains *how* he performed the computation (the standard algorithm), but does not draw on any mathematical coherency to do this. Also, Ralphie does not justify why he is performing $16 \times 8$. The problem is decontextualized, so no justification of this nature can be given other than, “Because the problem asked me to.” An objection might be raised that Ralphie does in fact justify his use of the standard algorithm by stating, “I don’t know my 16s very well.” However, this is not a *mathematical* basis; that is, Ralphie does not appeal to any mathematical reasoning or contextual features of the problem to justify his use of the standard algorithm. Hence, Ralphie’s comment supports only *computational strategies*.

Region 6 in the Venn diagram represents student actions that supported both *computational strategies* and *justification*. The actions in this region were computationally-focused justifications where students explained both *how* and *why* computations were performed. Emmett’s explanation of $9 \times 3/4$ displays this *how* and *why*:

Emmett: I got 6.75 because first what I did was I divided 9 by 4 to find out what 1/4 of 9 is. And then I multiplied that by 3 to figure out what 3/4 of 9 is.

Emmett justifies *why* he divided by 4, appealing to the relationship between fractions and the operation of division: dividing by 4 results in 1/4 of the original number. However, Emmett’s overall statement serves
to explain *how* he computed $9 \times 3/4$. Hence, Emmett evidences both *computational strategies* and *justification*. Other actions in Region 6 followed this pattern: students explained their computations in detail while also providing a basis to justify why their computations, or computational sub-steps, were valid.

Region 3 in the Venn diagram represents student actions that supported *justification* but none of the other sociomathematical norms. Actions in Region 3 consisted primarily of justifications that appealed to visual representations and justifications by contradiction. For example, Arnold asserted that the difference of two squares would never equal one. Keisha then challenged Arnold’s claim, exemplifying a justification by contradiction that falls in Region 3:

Arnold: 1 doesn’t work. You can’t make 1 [by taking a difference of two squares].
Ms. Skywalker: There’s no way you can make 1?
Keisha: Yes there is: $1^2 - 0^2$.

In challenging Arnold, Keisha implies an existence claim of her own: There exist two squares whose difference is 1. Keisha then provides a counterexample to Arnold’s claim that simultaneously serves as a justification for her own implied claim. Her justification (“$1^2 - 0^2$”) does not involve explaining how to perform a computation, establish mathematical coherency, or offer a different perspective. Therefore, it supports *justification* and not the other four sociomathematical norms. Likewise, many justifications that appealed to visual representations did not explain a computational strategy or establish mathematical coherency and so fell into Region 3 as well.

Region 5 at the center of the Venn diagram represents *multiple perspectives*. Its position at the central intersection of the other three sociomathematical norms represents its inextricable linkage with them. *Multiple perspectives* always occurred in conjunction with at least one of the other sociomathematical norms. This is because when students shared a different perspective, they necessarily had to share a different perspective of *something*: a different justification, a different computational strategy, or an equivalent way of conceptualizing or representing a situation. However, the position of
multiple perspectives at the center of the Venn diagram might inadvertently convey the impression that it always occurred in conjunction with all of the other three sociomathematical norms. This is certainly not the case and is a limitation of the Venn diagram representation given in Figure 4-8. To clarify, multiple perspectives always occurred with at least one other sociomathematical norm, but did not necessarily occur in conjunction with all three.

**Identified norms in light of the research questions**

This chapter has presented the findings from phase one of the study. Phase one was intended to address the first research question:

*What social and sociomathematical norms are associated with mathematically productive discourse?*

Recall that *mathematically productive discourse* was defined by three general norms: it holds mathematical reasoning as the authority, it focuses on sense-making, and it strives to create mathematical coherency. As explained in chapter 2, each of these three norms has been recognized to be of great importance to learning mathematics. Since these three norms are fairly abstract, the goal of phase one was to identify more specific, concrete norms that are associated with these definitional norms. I now consider the norms identified in the case study classroom in light of the three definitional norms of mathematically productive discourse.

The norm of coherency was evident in Ms. Skywalker’s classroom. As discussed earlier, there were five more concrete sub-norms that supported this general norm. These five sub-norms are listed below.

- Generalizing after observing patterns across multiple cases
- Identifying equivalencies across different notations, operations, units, and visual representations
- Recognizing when a particular problem manifests a more general mathematical property
- Transferring knowledge from a different problem
- Comparing numbers to landmark numbers to understand their relative size
Additionally, *multiple representations* supported the norm of coherency, as different representations, strategies, and explanations were some of the mathematical “objects” that students connected.

Holding mathematics as the authority means that students appeal to mathematical reasoning rather than non-mathematical reasons (such as the teacher, textbook, social status, etc.). In Ms. Skywalker’s classroom, mathematical reasoning was appealed to during justifications. Every time students justified their claims, they implicitly supported the idea that mathematical reasoning was the authority. As discussed earlier, *justification* was supported by four more concrete sub-norms:

- *Justifying by providing computational steps as well as a basis for those steps*
- *Justifying through coherency strategies*
- *Justifying through visual representations*
- *Justifying by contradiction*

Sense-making is the process by which students come to personally understand the meaning of the mathematics that they are doing. Unlike the other two definitional norms, there was not one norm in Ms. Skywalker’s class that most prominently supported sense-making. Instead, the norm of sense-making appeared to be supported by a combination of some of the norms and sub-norms discussed earlier. The *coherency* sub-norm of identifying equivalencies often supported sense-making, exemplified in the following dialogue as the class simplifies the improper fraction \(\frac{24}{3}\) to 8.

Ms. Skywalker:  What?
Janet:  Um, a fraction is basically dividing the numerator by the denominator. So \(\frac{24}{3}\) is the same as 24 divided by 3.

In this case, Janet personally understands the meaning of fraction notation through its equivalence with division. That is, she interprets the fraction notation via an equivalency. *Active listening* also helped to support sense-making. Students often made sense of mathematical topics and concepts after listening to their peers’ explanations. In the next example, Emmett has just explained how he added 5 weeks, 6 days to 4 weeks, 5 days, in the process grouping 7 days and adding them to the weeks (shown in Figure 4-9). After this sharing, Leah speaks up:
Ms. Skywalker: Yeah?
Leah: So he just added one week to the 5 and the 4 weeks because 6 plus 5 is 11 [days]. And that he knows that there’s 7 days in a week. And so he took one—he took 7 [days] away and then put it in [the weeks].

After listening carefully to Emmett’s explanation, Leah revoices it, indicating that she personally understood the meaning of Emmett’s computations. Hence, active listening helped to support Leah’s sense-making.

**Minor norms**

As explained at the beginning of this chapter, there were five “major” noteworthy norms identified in Ms. Skywalker’s classroom: active listening, coherency, justification, computational strategies, and multiple representations. These norms have been discussed and presented in detail. However, there were also two “minor” norms identified that were not classified as “major” norms because they lacked the strength of evidence: precision and metacognitive articulation. I will summarize both of these minor norms and explain why they did not achieve an equal standing with the five major norms.

For reference, Table 4-1 provides the numerical data for the number of student actions and student member check comments that supported each of these norms.
Table 4-1. Total number of student actions and student member check comments supporting each of the major and minor norms identified in Ms. Skywalker’s classroom.

<table>
<thead>
<tr>
<th>Norm Identified</th>
<th>Number of Student Actions</th>
<th>Number of Student Member Check Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coherency</td>
<td>112</td>
<td>9</td>
</tr>
<tr>
<td>Justification</td>
<td>102</td>
<td>6</td>
</tr>
<tr>
<td>Computational Strategies</td>
<td>62</td>
<td>12</td>
</tr>
<tr>
<td>Active Listening</td>
<td>60</td>
<td>4</td>
</tr>
<tr>
<td>Multiple Perspectives</td>
<td>44</td>
<td>5</td>
</tr>
<tr>
<td>Precision</td>
<td>24</td>
<td>0</td>
</tr>
<tr>
<td>Metacognitive articulation</td>
<td>9</td>
<td>3</td>
</tr>
</tbody>
</table>

**Precision**

_Students demonstrate precision in their language when talking about mathematics. This is evidenced through students using appropriate mathematical terminology_.

_Precision_ means that students’ talk about mathematics uses appropriate, precise mathematical terminology. An exemplar of this is Jessie’s explanation of how she performed $76 \times 89$. As Jessie explains the steps of the multiplication algorithm, note how she refers to the value that each digit represents, rather than the digit itself.

Jessie: I wrote $76 \times 89$… And then I did seventy times eighty equals 5,600. And then I did eighty times six and I got 480… And then I did seventy times nine equals 630. And then six times nine is 54. And then you add them.

Jessie’s reference to values rather than digits (e.g. “eighty” rather than “eight”) reflects linguistic precision as she discusses the multiplication algorithm. Likewise, note the precise terminology Arnold uses as he discusses the two shapes shown below in Figure 4-10:
Arnold: So to make sure it’s a square, you have to have right angles on all sides. But this [shape on the left] has only two right angles, an obtuse [angle], and an acute [angle]. But this shape [on the right] has four right angles. So if it has four right angles, it can either be a rectangle or a square. But if these two sides are the same length, the two adjacent sides are the same length, and it has four right angles, then it has to be a square.

As Arnold discusses the shapes, he employs precise terms such as right angle, obtuse, acute, and adjacent. Although there were many similar exemplars, there was also a reasonable amount of disconfirming evidence against this norm. An exemplar of this disconfirming evidence is Liz’s explanation of the differences between consecutive square numbers:

Ms. Skywalker: What are we saying though? What do we see? Come on, someone describe this pattern for me.
Liz: It goes up by 2 every time.

Liz is attempting to explain that the difference of two consecutive square numbers is 2 less than the difference of the next two consecutive square numbers (e.g. $4 - 1 = 3; 9 - 4 = 5$). However, her explanation lacks appropriate terminology and the referent of “it” is unclear. Rebecca’s explanation, given below, is another exemplar of disconfirming evidence. A table has been written on the board, shown in Figure 4-11. Figure 4-12 is a recreation for clarity’s sake. The top row of the table lists multiples of 12, while the bottom row lists multiples of 8.
Figure 4-11. The table containing multiples of 12 and 8 that the class was discussing.

<table>
<thead>
<tr>
<th>12</th>
<th>24</th>
<th>36</th>
<th>48</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>16</td>
<td>24</td>
<td>32</td>
<td>40</td>
</tr>
</tbody>
</table>

Figure 4-12. A recreation of the table containing multiples of 12 and 8.

Ms. Skywalker: Tell the person next to you [what you are seeing here].

... Ms. Skywalker: Rebecca?
Rebecca: Um, Leah told me that it starts with 1 and then on, on the first row, on the top row it has 1 in between. 1 and then 1 again and 1 again and then 2. And then 2 again and then it will probably go to 2 again.
Ms. Skywalker: Okay, can you show me where you’re talking about?

Rebecca’s generalization is so unclear that Ms. Skywalker cannot understand what she is saying. She asks Rebecca to come to the board and point at specific numbers in order to clarify what the pattern actually is. Afterwards, it became apparent that Rebecca was referring to the number of multiples of 8 and 12 between each common multiple: There is one multiple of 8 and two multiples of 12 between each common multiple. However, her generalization as-worded lacks essential terms such as “common multiple” and is unclear as a result. In addition to the disconfirming evidence, precision lacked support from both student and teacher member checks. Given this disconfirming evidence, it was not classified among the major norms.

**Metacognitive articulation**

Students articulate an awareness of their own level of understanding. This is evidenced through the following sub-norms:

- Articulating their own familiarity with different types of computational strategies
- Articulating when they’ve understood a new concept or term
Metacognition means that students are aware of their own thinking and level of understanding. 

*Metacognitive articulation* means that students explicitly articulate this self-awareness of their own thinking. A paragon for this is Janet’s explanation of how she computed $3 \frac{1}{5} \times 8$:

Ms. Skywalker: Janet, will you explain your thinking? How you started this problem?
Janet: Sure. I don’t know how to multiply fractions, but I *do* know how to multiply decimals.
Ms. Skywalker: What do we got?
Janet: And so I converted $\frac{1}{5}$ into $\frac{2}{10}$, or 3.2.

In her explanation, Janet demonstrates an awareness of her own familiarity with different types of notation: she is comfortable multiplying decimals, but not multiplying fractions. Metacognition was referred to a few students during member checks:

Dorothy Ann: Asking questions if you’re confused… so you can actually understand what’s going on.
Wanda: Actually knowing what you’re doing.

Although the word “metacognition” was not actually mentioned, both comments implicitly assume it. Dorothy Ann’s comment of “asking questions if you’re confused” assumes an awareness of whether or not confusion exists. Likewise, Wanda’s comment of “actually knowing what you’re doing” assumes an awareness of whether or not this is the case. Hence, these comments implicitly support the idea of metacognition. Nevertheless, *metacognitive articulation* was not included among the major norms. Although there was not disconfirming evidence per se, there was simply a dearth of confirming evidence, as indicated by Table 4-1. Or put another way, there were abundant opportunities where students could have spoken in a metacognitive fashion and didn’t. Consequently, *metacognitive articulation* was not included among the five major norms.
Chapter 5: Phase Two Results

The two main research questions that guided this study are as follows:

1. What social and sociomathematical norms are associated with mathematically productive discourse?

2. What strategies can a teacher use to establish these norms in their classroom?

The previous chapter addressed the first research question by describing the norms that were identified in the case study classroom. This chapter now addresses the second research question by describing the teacher strategies used in the case study classroom to establish the previously identified norms. Like the previous chapter, this chapter is descriptive in nature. An in-depth interpretation of the results, their implications, and their relationship to the research literature will be considered in the final chapter.

As discussed in chapter 3, data for identifying teacher strategies came from both observations performed during the initial two months of the 2014–2015 school year, as well as interviews with Ms. Skywalker. The first two months of the school year is the time period when most classroom norms are being established (McClain & Cobb, 2001; Wood, 1999; Wood, Cobb, & Yackel, 1991). Thirteen different class sessions were observed during these initial months for a total of approximately 20 hours of observation time. In analyzing this data, I looked specifically for teacher actions that supported the norms identified in phase one. “Teacher actions” encompasses the totality of what Ms. Skywalker did and includes physical actions (e.g. writing on the board, gesturing), discursive actions, and decisions made prior to class (e.g. selection of mathematical tasks). I did not try to identify and categorize all teacher actions, but only those that seemed to support the phase one norms. In addition to the observations, I conducted twelve different interviews with Ms. Skywalker throughout the 2014–2015 school year. Ms. Skywalker was eager to talk about her classroom and teaching philosophy at length. These interview data complemented and clarified the observational data, allowing me to gain a more complete picture of Ms. Skywalker’s teaching strategies.
Ms. Skywalker’s teacher strategies naturally divided into two categories: *micro strategies* and *macro strategies*. Micro strategies were more narrowly-focused and tended to exert a direct, immediate effect in promoting the intended norm. These strategies were employed by Ms. Skywalker in light of the immediate circumstances taking place within her classroom. Examples include direct prompts and normative comments. By contrast, macro strategies were made in light of the overall, long-term development of the class, rather than immediate circumstances. These strategies were generally employed for their long-term effects and tended to have an indirect effect in promoting the norms. Examples include selection and use of mathematical tasks as well as general expectations for the classroom environment. To clarify the distinction between micro and macro strategies, they may be considered as Ms. Skywalker’s strategies when viewed through a “wide-angle lens” and a “zoomed-in lens” respectively. Macro strategies correspond to the wide-angle lens because they are made in consideration of the class’s development over months. Micro strategies correspond to the zoomed-in lens because they are made in response to the immediate context. I will begin in the first half of this chapter by presenting Ms. Skywalker’s micro strategies in detail, before proceeding to discuss her macro strategies in the second half of the chapter. I will then close with a brief summary of how these results answer the original research question.

**Micro strategies**

The micro strategies that I identified used to establish the productive norms fell into four categories:

1) *Direct Prompts*. The teacher directly prompts (through either the form of a directive or a question) a student to immediately *do something* that is consistent with a norm (e.g. a prompt to justify, share another solution strategy, show a hand signal).

2) *Normative Comments*. The teacher talks *about* a norm in a general way that signals her expectations (e.g. the importance of justifying, the importance of sharing multiple solutions, etc.)
3) **Highlighting a Positive Example.** The teacher draws attention to a student who has acted in a manner consistent with a norm, either by praising the student or revoicing what the student has said or done.

4) **Modeling.** The teacher directly models behavior consistent with a norm (e.g. modeling a hand signal herself, explaining a computational strategy). Depending on the norm, this might include direct instruction (e.g. explaining a connection between two concepts).

Figure 5-1 details how frequently I witnessed Ms. Skywalker using each of these micro strategies for each norm during my observations of the first two months of the 2014–2015 school year. In the upcoming sections, I will systematically discuss, norm-by-norm, how the micro strategies were used to support each norm.

![Figure 5-1. The frequency of micro strategies used by Ms. Skywalker to establish each norm.](image)

**Active listening**

As Figure 5-1 shows, Ms. Skywalker overwhelmingly relied on direct prompts and modeling to promote *active listening*. This was due, in large part, to the hand signals that she expected her students to use. Recall that Ms. Skywalker taught her students how to indicate “yes” and “no” in American Sign
Language. The majority of the direct prompts for active listening involved Ms. Skywalker asking for hand signals in response to a yes-or-no question such as, “Who agrees with that answer?”, “Who used a similar strategy?”, or “Who understands what he just said?”. For example, in the following exemplar, students had just finished talking with their partners about a problem:

    Ms. Skywalker: Did talking with [your partner] help you a little bit? No? Show me, show me! Don’t call out. (*models hand signals for "yes" and "no")

These sorts of direct prompts for a hand signal were a common occurrence during the first two months. In addition to hand signals, Ms. Skywalker frequently utilized pair-share time, in the process deliberately asking students to share what their partner said. She emphasized this strongly and immediately corrected students if they forgot to share their partner’s thinking, as the following episode demonstrates:

    Ms. Skywalker: So why can [we] do addition when it’s really subtraction, and I do not want you to tell me what you said. I want you to tell me what your alpha partner said. What did your alpha partner say?
    Arnold: Yep, yep, yep, yep, yep, yep, yep, yep, yep, yep, yep, yep.
    Ms. Skywalker: Stop. That is not a reason. What’s the reason?
    Arnold: Well, I told him that you—
    Ms. Skywalker: Nah, nah, nah, nah!
    Arnold: He didn’t tell me, I told him.
    Ms. Skywalker: Then stop.

Together, the requests to share a partner’s thinking and the requests for hand signals accounted for most of the direct prompts for active listening. The remaining direct prompts occurred when Ms. Skywalker asked students to listen to their peers and identify similarities between strategies, either before or after those strategies had been shared. The following exemplar demonstrates this, as Jessie and Ralphie had just finished sharing their solution strategies with the class:

    Ms. Skywalker: Do you guys notice anything similar between what Jessie said and what Ralphie said? Do you? Did you hear anything that was similar between

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15 For an explanation of what “pair-share” time is, refer to chapter 4.
them? (models hand signals for “yes” and “no”)

Ms. Skywalker’s normative comments about active listening typically stressed that it would aid students’ learning. The following comment exemplifies this kind of remark:

Ms. Skywalker: Part of your job in learning is listening, isn’t it? Sometimes when I ask you to share with someone, I’m going to ask you what the person you talked to said, not what you said. Because your job when you’re talking to someone is not to wait for them to stop talking so you can say what you want to say… You can learn something from every single kid in this class.

Normative comments like these complemented the direct prompts by illuminating the rationale behind them as well as highlighting their potential benefits. Together, the direct prompts and normative comments encouraged students to actively listen to their peers while simultaneously understanding why such listening was important.

Ms. Skywalker did not rely on highlighting positive examples to support active listening as much as for other norms. Most of the highlighting that she did do came in the form of praise for students who shared their partner’s thinking without prompting, as in the following episode:

Ms. Skywalker: Jessie?
Jessie: My alpha partner said that 4/7 is like—
Ms. Skywalker: Stop. Did you hear that she told me what her alpha partner said? I didn’t even tell her to tell me what her alpha partner said. Go put two marbles in the “We Rock Jar” for her because she’s listening to other people in our classroom. Good listening!

Similar to the direct prompts, the majority of the modeling actions for active listening involved Ms. Skywalker modeling a hand signal herself, typically as she prompted her students to do likewise.

Coherency

Ms. Skywalker employed a relatively balanced mix of the four micro strategies to promote coherency. Her direct prompts demonstrated a great amount of variety. Sometimes, these direct prompts
were very general and open-ended. For example, on one occasion, Ms. Skywalker wrote multiples of 3 on the board in a 6 by 5 grid layout as shown in Figure 5-2 below.

![Figure 5-2. The multiples of 3 that Ms. Skywalker wrote on the board.](image)

After the class read the multiples of 3 out loud, she asked, “What do you notice?” This simple question was intended to prompt students to look for patterns in the grid and make generalizations without guiding or steering them in any particular way. Other direct prompts focused student attention more narrowly on a few mathematical objects while still refraining from guiding or steering them to any particular conclusion. An example of this more-focused prompt occurred when Ms. Skywalker wrote the factors and arrays for the number 15 on the white board, as shown in Figure 5-3 below. She then asked, “How are [the factors] related to [the arrays]?”.  

![Figure 5-3. The factors and arrays of the number 15 that Ms. Skywalker wrote on the board.](image)
This question focused students’ attention specifically on comparing the factors with the arrays. Hence, it was a more focused question than simply asking, “What do you notice?”. However, it still did not give any indication of how exactly the factors and arrays were related. On other occasions, Ms. Skywalker’s direct prompts for coherency clearly guided students towards a specific connection that she wanted them to recognize. An exemplar of this specific, guided prompt is the following episode:

Carlos: I don’t know why, but the hardest math problem [to remember] is 7 × 6.
Ms. Skywalker: That’s just hard for you to remember?
Carlos: Yeah.
Ms. Skywalker: So here’s a strategy: if 8 × 8 was hard for you, 8 × 4 is one of my automatics. I don’t know about you, but 8 × 4 is 32. What could I do with that knowledge to know 8 × 8?

In this situation, Ms. Skywalker wanted her class to recognize that doubling a factor results in doubling the product. To lead them towards this connection, she provided 8 × 4 and 8 × 8 as examples, implying the existence of an easy way to leverage 8 × 4 in computing 8 × 8. Another type of guided direct prompt was a press for students to extend a generalization. For example, to a group of students who had developed a generalization dealing with whole numbers, Ms. Skywalker asked, “What about decimals?”. Questions like these guided students to extend a generalization to a new domain. Finally, Ms. Skywalker also prompted students to establish coherency by challenging a valid claim in order to elicit a justification. For example, when a student answered, 4/8 or 2/4 or 1/2 to a relatively simple question, Ms. Skywalker responded in the following manner:

Ms. Skywalker: So what he’s saying is 4/8 = 2/4 = 1/2…Am I right, is this what he said? But I have to tell you something: 4/8 is not the same as 2/4. It’s not the same as 1/2. What do I mean by “same”? What does he mean by “same”? 4’s not the same as 2. Here’s 4, here’s 2. They’re not the same! Here’s 8, here’s 4, they’re not the same! So how is this the same?
In this situation, Ms. Skywalker’s challenge led to students discussing the concept of equivalent fractions and justifying why the fractions in question were equivalent. Thus, Ms. Skywalker’s challenge effectively functioned as a direct prompt for students to establish *coherency*.

Ms. Skywalker’s normative comments about *coherency* mainly centered around two themes. The first theme emphasized that interconnectedness was part of the very essence of mathematics. Comments of this type typically mentioned the reasonableness of mathematics, the numerous patterns that existed within it, and that pattern-finding was a significant part of “doing math.” The following normative comment exemplifies this theme:

Ms. Skywalker: This is what’s cool about math! There’s patterns, there’s logic, things make sense! They’re connected.

The second theme of Ms. Skywalker’s normative comments stressed the utility of *coherency* as a problem-solving strategy. These sorts of comments emphasized that making connections would help students transfer their current knowledge to solve new and unfamiliar problems. The following normative comment exemplifies this theme of transfer:

Ms. Skywalker: That’s why we’ve been doing all this thinking [about connections], so when [a problem] looks different you don’t shut down.

Ms. Skywalker also highlighted student actions that helped to establish *coherency*. Sometimes this highlighting meant praising the student’s actions. In the following episode, the class had just finished computing the perimeter of a regular octagon with side length 5 by performing $5 \times 8$. Ms. Skywalker then asked what the perimeter of the octagon would be if the side length was increased to 6. Several students indicated that computing $6 \times 8$ would yield the perimeter of this new octagon. At this point, Jessie spoke up and made a comment that indicated she had transferred knowledge from the first situation to the second:

Jessie: You could’ve done $5 \times 8$ and then you added 8.

Ms. Skywalker: Why can you do that?
Jessie: Because $5 \times 8$ is 40, but then $6 \times 8$, you’re just adding another 8.
Ms. Skywalker: Is it? Five 8’s and six 8’s? Just one more 8? And you’d already figured out five [8’s], didn’t you? I love how you used what we already did! You didn’t start over. You used what we already did. That’s a great skill, a great strategy. You guys see that strategy? I like that strategy.

Recognizing that Jessie had established coherency by relating the two problems, Ms. Skywalker praised Jessie’s response. In the process, she explicitly articulated the strategy that Jessie had used (“You used what we already did! You didn’t start over.”). This clarified the strategy for the rest of the class as well as the high esteem with which Ms. Skywalker viewed it. In other situations, Ms. Skywalker did not explicitly praise student actions that supported coherency, but highlighted them nevertheless through revoicing to draw the class’s attention to them. For example, in the midst of one problem, Arnold converted $8/9$ to a decimal by dividing 9 into 8. After he had finished explaining his work, Ms. Skywalker drew attention to the connection that Arnold had made between fractions and division:

Ms. Skywalker: He basically looked at this [fraction] as a division problem.

By revoicing part of Arnold’s work, Ms. Skywalker ensured that the entire class recognized the close relationship between fractions and division that Arnold had relied upon. By highlighting Arnold’s action, she thus promoted coherency in her classroom. In other situations, when students voiced reflection generalizations, Ms. Skywalker typically allowed time for the entire class to investigate and justify them. For example, when Quinton noticed that odd square numbers have odd square roots, Ms. Skywalker took several minutes to allow the class to investigate cases and then justify why this would always be true. This sort of investigation highlighted the generalization by making it the focus of classroom activity.

Finally, Ms. Skywalker was also comfortable modeling for her students how to establish coherency. Through this direct instruction, she pointed out connections between different mathematical topics, concepts, notations, and representations. For example, in the following exemplar, Ms. Skywalker explained the operation of division in terms of an array model for multiplication:
Ms. Skywalker: That’s all division is! Division is saying, “I’m going to give you the array… I’m giving you the area. But I’m only going to tell you what one side [length] is. You have to figure out the other side. That’s what division is!

Rather than introducing division as an entirely new and independent operation, Ms. Skywalker explained it in terms of familiar multiplication ideas, thus modeling for her students how to establish mathematical coherency.

**Justification**

In promoting justification, Ms. Skywalker primarily relied on direct prompts, as shown in Figure 5-1. These direct prompts overwhelmingly came in the form of a simple one-word question: Why? This question was so common in Ms. Skywalker’s class that her students soon came to expect it. There were several occasions when Ms. Skywalker asked her students, “What do you think I’m going to ask you?”, and the students quickly responded, “Why”. Ms. Skywalker used this one-word question to draw out justifications when students presented claims or answered questions. The following exemplar is representative of the many times this occurred:

Carlos: A polygon is a 3 or more sided shape that—
Ms. Skywalker: Why does it have to be 3? I don’t understand, why does it have to be 3?
Why? You’re right, so don’t let me fool you. You’re right! But why?
Carlos: I have no idea why.
Ms. Skywalker: Yes you do!

Ms. Skywalker demonstrated remarkable tenacity in continuing to press for a justification when students didn’t initially respond to or understand her initial prompts. She did not shy away from continuing to ask “Why?” and reframing the question without revealing the answer, so that students could understand it.

The following paragon shows one of Ms. Skywalker’s longer presses for justification. Most of her presses for justifications were shorter than this particular episode, not because she gave up, but simply because she elicited a satisfactory justification. This episode demonstrates how persistent Ms. Skywalker was willing to be in pressing for a justification when it was not provided. In the episode, she wanted her
students to justify why the number 15 is odd. As part of this justification, she wanted them to explicitly articulate why the ones place (rather than the tens place) determines the parity of the number:

Class: Odd!
Ms. Skywalker: So everyone thinks it’s odd. And now we need to know why you think it’s odd? I need some good whys.
Janet: 5 is odd.
Ms. Skywalker: 5 is odd. Keep going!
Janet: 5 is odd.
Ms. Skywalker: But this isn’t the number 5, this is the number 15.
Janet: I know. 5 is odd, so 15 is odd, because it ends in 5.
Ms. Skywalker: Why? I don’t understand.
Janet: Okay so, 5 is odd and 15 is odd. 15 is odd because it has 5 as the last number, and 5 is not divisible by 2.
Ms. Skywalker: Okay, but this is 15.
Janet: I know! 5 is not divisible by 2. The 10 doesn’t matter!
Ms. Skywalker: Why not?! Why doesn’t the 10 matter? I think it’s kind of important, without it, it’s not 15!
Janet: But it doesn’t matter when you’re deciding if—
Ms. Skywalker: Why doesn’t it matter? I know you know. Why doesn’t the 10 matter? Why can I just focus on the ones place? Why can I just focus on the ones place? Rebecca?
Rebecca: Umm… is it because? Uh….
Ms. Skywalker: You got it. You were going to tell me something about the tens place, weren’t you? What about it?
Rebecca: That because when you have a 10 it, if you have a 20 or a 10, any number that ends in a zero, zero is odd or even, but if you have 15—
Ms. Skywalker: 15 is a 10 and a 5.
Rebecca: Yeah, so it’s odd in the 5 spot, so when you add an odd number to the odd…
Ms. Skywalker: Is she good? She is so close! I get what she’s saying. Your thinking is good, girl. Matt!
Matt: Okay, so how I figured it out is: 10 was even, because even—
Ms. Skywalker: Right! Your saying 10 is even. Aren’t you? And 20 is even. And anything you put in the tens place is…
Class: Even.

Note in this paragon the number of times Ms. Skywalker continued to press for justification without giving away the answer through hints. This aptly demonstrates her remarkable tenacity in pressing for student-voiced justifications. In addition to repeated pressing, Ms. Skywalker also occasionally prompted justifications by challenging student claims, eliciting the justification as a response to the challenge. She did exactly this in the following exemplar as she challenged a student’s claim that $4/8 = 2/4 = 1/2$:
Ms. Skywalker: So what he’s saying is $4/8 = 2/4 = 1/2$... Am I right, is this what he said? But I have to tell you something: $4/8$ is not the same as $2/4$. It’s not the same as $1/2$. What do I mean by “same”? What does he mean by “same”? $4$’s not the same as $2$. Here’s $4$, here’s $2$. They’re not the same! Here’s $8$, here’s $4$, they’re not the same! So how is this the same?

By challenging the claim that $4/8 = 2/4 = 1/2$, Ms. Skywalker successfully elicited a justification of why two fractions are equivalent. In addition to eliciting justifications, Ms. Skywalker also frequently used direct prompts to draw out additional details within a justification. The next exemplar illustrates this phenomenon:

Carlos: All even numbers are composite except for $2$.
Ms. Skywalker: What do you guys think? Are all even numbers composite except for $2$? (multiple students respond with “yeah”)
Ms. Skywalker: Why? Why Tim?
Tim: Because all of them are even and they all have different numbers other than $1$ and itself that go into them.
Ms. Skywalker: Okay. And what is that one number that we know for sure goes into all of them?
Tim: $2$.

Notice how Ms. Skywalker first asked “why” to elicit a justification. Once the justification had been provided, she directly prompted again (“What is that one number that we know for sure goes into all of them?”) to explicitly draw out an additional detail of the justification. Similarly, Ms. Skywalker also used these sorts of follow-up prompts to draw out students’ bases for performing computational steps. Note how in the following exemplar, she drew out the basis for Quinton’s computation of $18 \times 38$:

Ms. Skywalker: How did you figure this out, Quinton?
Quinton: I did $18 \times 38$ which equals—
Ms. Skywalker: You did $18 \times 38$?
Quinton: Yes.
Ms. Skywalker: Why?
Quinton: And $9 \times 1$ is $9$ because—
Ms. Skywalker: Wait, wait, wait! Why did you do that?
Quinton: Because you go up by $18$ [for each column on the table].
This sort of direct prompt, to draw out the basis behind computational steps, was commonly employed by Ms. Skywalker.

Although Ms. Skywalker relied most frequently on direct prompts to elicit justifications, she also used normative comments about justification. Through these normative comments, she emphasized two main ideas. The first was her expectation for justification in all circumstances. These comments generally sounded similar to the exemplar below:

Ms. Skywalker: You guys! It’s not all about just getting the right answer! I want to know how you came up with it. What was your thinking? What was your strategy? Draw some pictures.

The second main idea conveyed through normative comments dealt with the *quality* of justification. On the first day that I observed, September 4th, Ms. Skywalker taught her class about different levels of justification. In the quote below, she distinguished between justification based on an authority figure and justification based on mathematical reasons:

Ms. Skywalker: Sometimes we will say things and we’ll say our reason for knowing something. And her reason for knowing something, and her reason for knowing this was an authority figure, right? “My teacher said. My babysitter said.” And that’s called… an authoritative or an authoritarian justification, right? You’re justifying what you know based on what an authority figure told you, right? That’s kind of a first-level of justification… but wouldn’t you rather know something because you figured it out and you have a really good reason?

Ms. Skywalker ended up distinguishing between three different types, or levels, of justification for her students: justification based on an authority figure, justification based on examples, and justification based on mathematical reasons. She would periodically refer back to these different levels when prompting for justification. For example, when asking her class to justify why some numbers are even and others are odd, she added the following clarification:

Ms. Skywalker: And don’t tell me, “Because Ms. Skywalker said.” Because that’s level one justification. I want some higher justification going on here, folks. Let’s see it.
These comments, along with others, served to reinforce the notion that an acceptable justification in Ms. Skywalker’s class meant appealing to mathematical reasons rather than citing an authority figure or a handful of examples.

Finally, in addition to direct prompts and normative comments, Ms. Skywalker also highlighted student-voiced justifications. This highlighting usually meant revoicing a student-voiced justification. In doing this, Ms. Skywalker would elaborate and add details in order to further elucidate the reasoning underlying the justification. For example, Rebecca justified that 15 was composite by pointing out that it had four factors. Ms. Skywalker then revoiced this justification, further clarifying it:

Ms. Skywalker: So this is what my friend Rebecca said, and she’s right. She’s saying… 15 is composite because it has four factors: 1, 3, 5, and 15. And prime numbers have exactly two factors.

Note the details and clarifications that Ms. Skywalker added to Rebecca’s justification. Rebecca had originally only stated in her justification that 15 had four factors. Ms. Skywalker clarified what those four factors were and explicitly mentioned the definition of prime to show that the number 15 did not fit this definition. Hence, she explicated reasons that were implicit in Rebecca’s justification. On a few occasions, Ms. Skywalker highlighted positive examples, not by revoicing, but by praising the students involved. For example, in the following exemplar, Arnold produced a reflection generalization about multiplicative parity properties. Several students immediately prompted Arnold with “Why?”. Ms. Skywalker then enthusiastically praised these students:

Arnold: If [a multiplication problem] has an even number [for a factor], [the product] is going to be an even number. But if both of [the factors] are odd, then [the product] will equal an odd number.

Ms. Skywalker: I like how you guys are all asking why! I’m not even asking why anymore! Y’all are!
Through this sort of praise, Ms. Skywalker highlighted the positive examples of students prompting each other for justification.

Notably, I never once observed Ms. Skywalker using modeling or direct instruction to provide a justification herself. While she used this micro strategy with all of the other identified norms, she never used it for justification. Rather than justifying herself, she would consistently prompt her students to justify. If they struggled or didn’t initially respond, she would persist in pressing for a justification, sometimes providing hints or basis elements (e.g. definitions) so that students would have the necessary information to justify. Even if this process took a long time, Ms. Skywalker never once “gave up” and did the justifying herself. Once a justification was produced, she showed no qualms about further explaining and elaborating upon it, but she insisted that the initial justification be student-voiced.

**Computational strategies**

As with coherency, Ms. Skywalker employed a relatively balanced mix of micro strategies to promote computational strategies. She frequently used direct prompts to get students to share their computational strategy, particularly when they had only volunteered an answer. This is what happens in the exemplar below, as Jessie initially shared only her answer to the computation $79 \times 89$:

```
Ms. Skywalker: Jessie, what did you get [for $76 \times 89$]?
Jessie: I got 6764.
Ms. Skywalker: How did you do this, Jessie?
Jessie: I used the traditional math.
Ms. Skywalker: You used the standard algorithm?
Jessie: Yeah.
Ms. Skywalker: Can you come up [to the board] and show me? Can you kind of tell us what you’re doing as you do it?
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In this incident, Ms. Skywalker prompted Jessie twice to explain her computational strategy: first to share what her strategy was (“How did you do this?”), and second to share the details of the strategy (“Can you kind of tell us what you’re doing as you do it?”). Many of Ms. Skywalker’s direct prompts for computational strategies bore similarities to this. Sometimes, Ms. Skywalker had a particular
computational strategy in mind that she wanted students to recognize. In these cases, she prompted
students by hinting at the strategy or by providing an additional piece of information to steer students
towards the strategy. In the following exemplar, she wanted students to transfer knowledge from a
similar, more familiar problem. Notice how she steered Carlos towards this idea:

Carlos: I don’t know why, but the hardest math problem [to remember] is $7 \times 6$.
Ms. Skywalker: That’s just hard for you to remember?
Carlos: Yeah.
Ms. Skywalker: So here’s a strategy: if $8 \times 8$ was hard for you, $8 \times 4$ is one of my
automatics. I don’t know about you, but $8 \times 4$ is 32. What could I do
with that knowledge to know $8 \times 8$?

By introducing additional information to this situation ("$8 \times 4$ is 32"), Ms. Skywalker guided Carlos
towards recognizing the strategy of transferring knowledge. In effect, Ms. Skywalker prompted Carlos to
use the strategy she had in mind without directly telling him what it was.

Ms. Skywalker did not utilize normative comments as frequently for computational strategies as
some of the other productive norms, as Figure 5-1 shows. About half of the normative comments that she
did make focused on using estimation before computing in order to gain an approximate sense of the
answer. The following remark, made to her students, exemplifies her comments of this type:

Ms. Skywalker: I want you to think in your head right now, I want you to make a
prediction… Do you know that you should do predicting all the time?
About how big of a number am I going to get? How reasonable is this
answer going to be? You do it all the time. We start all math by thinking.

Her other normative comments promoting computational strategies generally focused on a particular
computational strategy that had just arisen in whole-class discourse.

To highlight student-voiced computational strategies, Ms. Skywalker most frequently revoiced
them. During this revoicing, she would elaborate and add details in order to further elucidate the strategy
for the class’s benefit. For example, when the class was trying to compute $180 \div 26$, Arnold began with
26 and doubled it. To determine how many groups of 26 were in 180, he added multiples of 26 and 52
together until he reached 180. However, he explained quickly and several students indicated that they had not understood it. To help the class understand, Ms. Skywalker revoiced Arnold’s strategy, writing groups of 26 and 52 on the board, as shown in Figure 5-4. She then labeled in blue the cumulative number of groups of 26. These added details allowed the class to grasp and appreciate Arnold’s strategy.

Figure 5-4. Ms. Skywalker’s writing as she revoiced Arnold’s strategy for computing $180 \div 26$.

Ms. Skywalker also drew attention to students’ computational strategies by praising them, often in conjunction with revoicing. For example, in the following exemplar, Dorothy Ann explained her strategy for computing $12 \div 2$, after which Ms. Skywalker revoiced and praised her:

Dorothy Ann: I thought… “What plus what equals 12?” And then I remembered that $6 \times 2$ equals 12, and that’s the same as just adding 6 two times.

Ms. Skywalker: She used like two strategies there, didn’t she? Did you hear that? Two quick strategies in her head. Like, maybe even more than that. Doubles, multiplication [as] repeated addition, multiplication as the inverse operation of division. Right? Dang! Good stuff, girl!

In this situation, Ms. Skywalker quickly summarized the pertinent mathematical ideas that Dorothy Ann’s computational strategy relied upon before then praising Dorothy Ann herself. This episode reflects how Ms. Skywalker typically drew attention to student-voiced computational strategies. A less-common method that Ms. Skywalker used to draw attention to certain computational strategies was to “challenge”
valid strategies. For example, when the class was discussing how to solve for $y$ in the equation $15 - y = 9$, Jennifer explained how she used addition to determine $y$. To highlight the use of inverse operations in this case, Ms. Skywalker “challenged” Jennifer’s strategy:

Jennifer: [I got] 6... because $9 + 6 = 15$.
Ms. Skywalker: Okay, so that’s what you used. You used addition. Did you see that? She used addition. Anybody else use addition? Great strategy, but this is a subtraction problem, sweetie. You can’t use addition on a subtraction problem.

(multiple students chime in, “Yes you can!”)

By challenging a valid strategy, Ms. Skywalker effectively drew the class’s attention to it, leading to a brief discussion about the use of inverse operations as a computational tool. Finally, on several occasions, Ms. Skywalker drew attention to various computational strategies by listing them on the white board. Figure 5-5 shows an instance of when she did this. By listing computational strategies that students had used, she reminded the entire class of them and helped to promote their use.

Ms. Skywalker seemed to be the most comfortable using modeling and direct instruction to promote computational strategies than with any other sociomathematical norm. She often shared her own personal preference for certain computational strategies with her students, as in the exemplar below where she shares her preference for leveraging inverse operations to perform subtraction:
Ms. Skywalker: Can I tell you something? I use addition all the time for subtraction because I’m better at it. Right? That’s just the way my brain works. I immediately think about that.

Ms. Skywalker also used direct instruction to model and expound various computational strategies. In the following exemplar, she modeled the use of the distributive property as a strategy to perform $23 \times 7$. Her corresponding work on the board is shown in Figure 5-6:

Ms. Skywalker: The distributive property is really useful! … What if I wanted to do $23 \times 7$? That’s harder for me. I don’t know my 20s in my head. I could say this is the same as $7 \times (20 + 3)$… This is the same as $(7 \times 20) + (7 \times 3)$. You can distribute the 7.

Figure 5-6. Ms. Skywalker’s work when modeling how to use the distributive property as a computational strategy.

Sometimes, as in the example just provided, Ms. Skywalker’s direct instruction came in the form of telling information. In other situations, it came in the form of numerous, leading questions that required little thinking on students’ parts. In the following exemplar, the class was discussing how to perform $0.73 - 0.32$. Ms. Skywalker wanted her students to use inverse operations and “add up” from 0.32 to 0.73. Note the frequency of her questions and the extent of thinking they required:

Ms. Skywalker: What about [the strategy we used yesterday]? … Did we use an inverse operation? So this is subtraction. Could we start adding to 0.32 to get 0.73? Could we do that? What would we do first? … So the first thing [we’re]
going to do is [we’re] going to add 0.08… And now we have?

Carlos: 0.40.
Ms. Skywalker: And now let’s add 0.30 to it to get?
Carlos: 0.70.
Ms. Skywalker: And then what do we want to add to it?
Carlos: 0.01.
Arnold: No, 0.03!
Ms. Skywalker: Let’s see what we get. Do you guys see how we added up right here?

Technically, Ms. Skywalker asked numerous questions and elicited student involvement in performing the computation. However, all of her questions were either rhetorical or required minimal thinking. In essence, she was using direct instruction to “tell” her students what to do. Her questions functioned as more of a check for attention, rather than requesting meaningful input from students. Ms. Skywalker also utilized direct instruction to proactively alert students to common mistakes and misunderstandings before they actually happened. In the following exemplar, she alerted students to the common problem of ignoring place value when performing the standard multiplication algorithm. To illustrate this, she used the computation of $89 \times 76$ as an example:

Ms. Skywalker: Do you know what makes me nervous about this whenever I teach 5th graders [the standard multiplication algorithm]? This is what I hear them saying… tell me if you can identify what my problem might be with this: “6 $\times$ 9 is 54, put down your 4, carry your 5. 6 $\times$ 8 is 48, and 5 more is 53. Add a zero. 7 $\times$ 9 is 63. Put down your 3, carry your 6…

... Jessie: You said 6 $\times$ 8, not 6 $\times$ 80.
Ms. Skywalker: Oh, did you hear that? Because what the traditional algorithm does is it takes you away from your number sense into the thought of just digits. So I looked at this as just digits, right?

In this situation, Ms. Skywalker used modeling and direct instruction to illustrate a common misunderstanding about a computational strategy.

**Multiple perspectives**

Of the four micro strategies, Ms. Skywalker overwhelmingly relied on direct prompts to promote *multiple perspectives*. These prompts would most commonly come after a given student had finished
sharing an answer or solution strategy. Ms. Skywalker would then typically ask if another student wanted to share their strategy, answer, or idea. The following instance exemplifies these sorts of prompts:

Ms. Skywalker: Anyone want to refine that, or do it differently?

Notice how this prompt was fairly general and open-ended. It was not aimed at any one student in particular and was not necessarily requesting a certain type of perspective; students were invited to contribute something along the same lines as before (“Anyone want to refine that?”) or along different lines (“Or do it differently?”). This was most often the form that Ms. Skywalker’s direct prompts for multiple perspectives took. It must be stressed that a different “perspective” could encompass a number of different objects. Depending on the context, these sorts of direct prompts were calls for a different solution, solution strategy, computational strategy, visual representation, justification, generalization, or notation. Also, while Ms. Skywalker’s direct prompts were most often aimed at the class in general, they were occasionally more focused on particular individuals. For example, in the following exemplar, Ms. Skywalker had a particular student’s work in mind when prompting for an additional perspective:

Ms. Skywalker: Somebody else had another way of saying [why 15 is odd], I saw it [on their paper], why [15] is odd. Who was it? Was it—
Ralphie: I think it was Biff.
Ms. Skywalker: It was Biff. Tell me why [15] is odd, Biff.

Ms. Skywalker’s direct prompts for multiple perspectives also occasionally called for hypothetical perspectives; that is, they were not calling for a particular perspective that students had actually used, but for a particular perspective that students could have used in a particular situation. In the following exemplar, Ms. Skywalker prompted Valerie, who was unsuccessful in completing a task, for a strategy that she could have used:

Ms. Skywalker: Valerie, what could you have done to figure this out now that you’ve seen this chart. What could you have done?
Thus far, all of the direct prompts considered have been prompts for *multiple perspectives* on a classroom level. In other words, the *class* offered multiple perspectives as different *individuals* contributed their single perspective. Almost all of Ms. Skywalker’s direct prompts for *multiple perspectives* fell into this category. She solicited multiple perspectives from the class as a whole, rather than from any single individual. Occasionally however, Ms. Skywalker did prompt individuals to employ multiple perspectives. In the following instance, she asked each student to create ten different expressions that were equivalent to the number 22:

**Ms. Skywalker:** I would like you to stretch yourself. You have 10 representations [to make of the number 22]. I want one to have a decimal in it, and I want one to have a fraction in it, and I want one to have a power. That make sense?

Thus, while most of Ms. Skywalker’s direct prompts for multiple perspectives were on a classroom level, a minority of them were on an individual level.

Ms. Skywalker did not make as many normative comments regarding *multiple perspectives* as with other norms, as shown by Figure 5-1. However, the normative comments that she did make tended to emphasize the utility of *multiple perspectives* in handling various problems. For example, in the following instance, she drew attention to the usefulness of *multiple perspectives* in allowing the class to compare the fractions $1/3$ and $1/4$:

**Ms. Skywalker:** If we want to compare two fractions, if we want to represent these fractions [$1/3$ and $1/4$], that’s why we do multiple representations, right? So if we represent $1/4$ in twenty-fourths and $1/3$ in twenty-fourths, is that easier for you to compare? … So there’s a reason why we need multiple representations of things sometimes, right?

Her other normative comments followed a similar pattern of emphasizing the usefulness of *multiple perspectives*, with the particular situation at hand being used as an example.

Compared to the other norms, Ms. Skywalker did not frequently highlight student use of multiple perspectives. There were only a handful of times when she praised or revoiced the use of them. This is likely because the majority of the time, Ms. Skywalker elicited multiple perspectives from her class as a
whole, rather than from a particular individual. However, her praise and revoicing tended to focus on *individuals* rather than her class as a whole. In other words, Ms. Skywalker would solicit multiple perspectives from her class as various individuals each shared their single perspective. As this happened, she would praise or revoice *individuals’* perspectives, rather than praising or revoicing her *entire class* for the diversity of perspectives. Consequently, relatively little praise or revoicing focused explicitly on the use of multiple perspectives. The handful of times that Ms. Skywalker did highlight multiple perspectives occurred because of an *individual’s* use of them. This was what happened in the following exemplar:

Ms. Skywalker: If the measure of this interior angle is 130 degrees, I want you and everyone else to tell me how I could figure out what the measure of this exterior angle is...

Liz: We thought of two different ways.

Ms. Skywalker: Okay, two different ways. I love it! … I love how you guys are doing multiple strategies, multiple representations. Yay! Good for you! If you did that in your group, pat yourself on the back and say, “We are super stars!”

Although Liz worked as part of a group to develop two different strategies for measuring the exterior angle in question, she presented these strategies to Ms. Skywalker as an individual. Thus, Ms. Skywalker explicitly praised the use of *multiple* strategies rather than just the use of a strategy.

Ms. Skywalker used very little direct instruction or modeling to promote *multiple perspectives.* This was likely because very little modeling was necessary. Simple, direct prompts for additional perspectives were sufficient to elicit them. Even from the beginning of the school year, students appeared to understand what these prompts were requesting and responded to them appropriately. Most of the times that Ms. Skywalker did model *multiple perspectives* occurred during the class’s investigation of non-standard multiplication algorithms. In these situations, Ms. Skywalker purposefully introduced the various algorithms through direct instruction and then allowed her students to further investigate why the algorithms worked.
Macro strategies

Interviews with Ms. Skywalker, as well as observations, revealed that her teacher strategies were not limited to narrowly-focused micro strategies. She also had a more general, overarching plan for her class’s mathematical development over the initial weeks and months of the school year. The collection of broader teacher strategies that comprised this plan I have deemed macro strategies. Unlike the micro strategies, macro strategies did not directly support the productive norms per se. Rather, they indirectly supported these norms through establishing an overall conducive environment for the norms to emerge, equipping students with relevant tools and skills to help them practice the norms, and creating frequent opportunities for the norms to be expressed. In an interview on September 11th, Ms. Skywalker indicated that these sorts of broad, macro strategies were necessary during the beginning of the school year in order to establish a foundation for the class to build on. Although Ms. Skywalker spoke specifically about the Common Core mathematical practices in the following quote, she revealed much about her general philosophy for the beginning of the school year:

Ms. Skywalker: The other thing [I’ve been wanting to do] is the idea of making the Common Core math practices more explicit to [the students]… There’s several things we need to do first. But I think we’re building towards that… You can’t just come in do these [Common Core math practices]. I mean, you can understand them all you want, but there’s some foundation things that need to happen [first]… You cannot have productive mathematical discourse… you can’t do any of those things without some foundation skills, if you will, by your kids.

In this quote, Ms. Skywalker delineated between a finished product (e.g. Common Core math practices) and the process of working towards that product. She recognized that merely having a clear vision or understanding of the end goal (“You can understand them all you want”) is insufficient for establishing that goal. Instead, she viewed the beginning of the school year as a process, recognizing that more complex mathematical practices required more basic, “foundation things” or “foundation skills” to be

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16 I had not used the term “productive mathematical discourse” in my interviews with Ms. Skywalker, so she likely did not use this term to indicate the exact qualities that I do. For my definition and explanation of productive mathematical discourse, refer to chapter 2.
present first. Since many of the Common Core mathematical practices are similar to the productive norms outlined in phase one, it is reasonable to conclude that many of the “foundation things” that Ms. Skywalker has in mind here are relevant to the establishment of the productive norms as well. In the upcoming sections, I will systematically introduce and explain the “foundation things” that Ms. Skywalker relied upon to establish the four productive sociomathematical norms. For explanatory purposes, I have decomposed these “foundation things” into three macro strategies: conducive classroom environment, foundational skills, and concept-oriented task philosophy. The conducive classroom environment strategy helped to establish an overall, general environment that supported the productive sociomathematical norms. The foundation skills strategy meant that Ms. Skywalker taught her students specific skills that allowed them to more effectively practice the productive sociomathematical norms. Finally, Ms. Skywalker’s concept-oriented task philosophy yielded frequent opportunities for students to practice the sociomathematical norms. Each of these macro strategies contains sub-strategies, which will also be elaborated upon. In the upcoming sections, I will draw on interview statements from Ms. Skywalker as well as observational data to support my claims. After all this, I will present a model which visually represents Ms. Skywalker’s macro strategies. Finally at the end of the chapter, I will reflect on both the micro and macro strategies in light of the original research questions. For clarification, I reiterate that the labels of conducive classroom environment, foundational skills, and concept-oriented task philosophy were not introduced by Ms. Skywalker. Rather, they were my creation for the purpose of organizing and explaining the macro strategies that I witnessed.

**Conducive classroom environment**

Ms. Skywalker seeks to create a classroom environment that is conducive for the productive sociomathematical norms to emerge. She does this by creating a classroom where:

- **Learning is exciting and fun**
- **Students feel safe**
- **Understanding is emphasized over performance**
- **The teacher knows individual students’ strengths and weaknesses**

17 I say “four productive sociomathematical norms” here because in Ms. Skywalker’s mind, active listening was part of a larger macro strategy. In other words, Ms. Skywalker used active listening as a means to establish the sociomathematical norms. This will be explained in more detail later on in the foundational skills section.
- **Helpful logistics and procedures are established**

Ms. Skywalker’s first macro strategy was to create a *conducive classroom environment* for the productive sociomathematical norms to emerge. By calling this the “first” macro strategy, I mean that this is the first macro strategy that I will discuss and *not* that this strategy occurred chronologically prior to the others. In the September 11th interview, when Ms. Skywalker talked about the process of working towards the Common Core practices, she referred to certain environmental factors as foundational. This has been italicized in the transcript below:

Ms. Skywalker: The other thing [I’ve been wanting to do] is the idea of making the Common Core Math Practices more explicit to [the students]… There’s several things we need to do first. But I think we’re building towards that… You can’t just come in do these [Common Core Math Practices]. I mean, you can understand them all you want, but there’s some foundation things that need to happen [first]. And I think one of them is… *an environment where learning is exciting, it’s fun, it’s safe, and you have rich activities and there’s certain systems and procedures in place to get started that way. I think you need to know your students…* I think all of those are foundation skills before you can do [the Common Core Math Practices]. You cannot have productive mathematical discourse… you can’t do any of those things without some foundation skills, if you will, by your kids.

The italicized parts of the transcript indicate that, in Ms. Skywalker’s mind, the overall classroom environment is a key part of the foundation she wishes to establish at the beginning of the school year. I have decomposed this macro strategy into five sub-strategies, which will now be discussed in turn. Afterwards, I will elaborate more specifically on how these sub-strategies supported the productive sociomathematical norms.

The first sub-strategy of creating a *conducive classroom environment* is for learning to be exciting and fun. Ms. Skywalker mentioned this in the September 11th interview quoted in the preceding paragraph. Observational data frequently showed Ms. Skywalker commenting on how math was fun, interesting, and exciting. The following exemplars are just two of many similar comments of this nature that occurred during whole-class discourse:
Ms. Skywalker: What feels great is when you figure [a math problem] out, huh? That feels awesome!

Ms. Skywalker: Isn’t this fun? Who’s having fun? Well it’s math, of course it’s fun! Good grief!

As an energetic and effervescent personality, she typically made these sorts of comments with great fervor and conviction. From both observations and interviews, it was plain that Ms. Skywalker genuinely enjoyed math and wanted her students to as well. She enthusiastically praised their work, their ideas, their strategies, and different perspectives they offered on the same problem. She talked about the satisfaction that comes from learning, from justifying, and from making connections. She spoke frequently of mathematics itself being useful, enjoyable, interesting, and related to the surrounding world. Ms. Skywalker made these sorts of comments both in whole-class discourse and to individual students as she circulated the room. In the following exemplar on September 4th, Ms. Skywalker talked individually with a student who wasn’t sure if she liked math. Notice how Ms. Skywalker conveyed her own enthusiasm for the subject:

Ms. Skywalker: Has math not seemed very fun [in the past], maybe? It’s going to be a lot more fun this year because math is actually really fun… Have you ever had that feeling [when] you understood something? Do you remember what that feeling’s like? Isn’t it good? It’s good, isn’t it? You can get that in math every day. Seriously! That’s my favorite part about it.

In this particular instance, she specifically mentioned the satisfaction that comes from learning. The tasks that Ms. Skywalker chose also encouraged a fun and exciting classroom environment. In the following interview transcript, she explained the importance of rich mathematical tasks18 that allowed for a sense of discovery:

Ms. Skywalker: I think that rich tasks and things like that, [the students] enjoy what they’re doing, and it’s fun.

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18 In the professional development that Ms. Skywalker participated in, a “rich mathematical task” was defined as a complex, non-algorithmic, non-routine task that allowed for multiple strategies and no single, clear-cut solution path.
Peter: There’s that sense of discovery and you can see they get into it.
Ms. Skywalker: Yeah, and that feels good. That’s what we forget: that feels good. Sitting and struggling through a math worksheet doesn’t feel good for anyone.

Ms. Skywalker firmly believed that repeated drill was not beneficial for students’ enjoyment of mathematics. Observational data confirmed that she consistently selected and used tasks in a way that focused on concepts and discovery, rather than repeated drill. This will be discussed in detail in the concept-oriented task philosophy section. For now, it suffices to say that Ms. Skywalker used tasks to help support her goal of creating a fun and exciting classroom environment.

The second sub-strategy of creating a conducive classroom environment is to create a safe environment for students. In this context, “safe” refers not primarily to physical safety (although that certainly is important), but to safety from ridicule and humiliation. Through interviews, Ms. Skywalker reiterated the importance of this kind of safety, as the following two quotes indicate:

Ms. Skywalker: One of [the foundation things to enable the Common Core math practices] is… an environment where learning… is safe.

Ms. Skywalker: Huge! [Safety] is huge! It really is.

Ms. Skywalker recognized that every time a student shared an idea in front of peers, they were taking a risk of potential embarrassment. If students were humiliated, then they would cease participating in whole-class discourse and would no longer be willing to share their ideas. Once this reluctance had set in, Ms. Skywalker doubted it could ever be overcome. Furthermore, she recognized that her own words of either encouragement or criticism carried tremendous weight for the students. In the interview quote below, she explains this influence:

Ms. Skywalker: [When students share their reasoning], their self-esteem is out there, it’s a risk… I can make them feel stupid, I have that power. Isn’t that sad? But I also have the power to make them feel good, and make them feel like [they’re] getting something… Kids will do almost anything not to look dumb.
At this point, the question arises: how exactly did Ms. Skywalker use her “power” to make her students “feel good, and make them feel like they’re getting something”? During interviews, she repeatedly used the phrase “honoring student thinking.” As she elaborated, it became apparent that “honoring student thinking” meant the norm of *multiple perspectives*. In the two interview quotes below, Ms. Skywalker never used the exact phrase “multiple perspectives,” yet she articulated the basic idea as she talked about creating a safe environment:

Ms. Skywalker: Part of that whole “safe” thing is: honoring multiple representations and strategies and learning styles. I think it starts by saying, “We’re all different. We all have strengths.” And I think that that’s one of those underlying skills that we don’t talk about, so that when someone does put up a representation, people out here aren’t saying, “Oh my gosh, that’s the *elementary* representation,” or “That’s not a great representation.” But it goes back to what you said before: that’s not a safe environment. That’s not a risk-free environment. So to create [a safe environment], you have to honor all children and honor all learning styles, and then you start to honor all representations. And that builds.

Ms. Skywalker: All thinking is important, right? That all of those [student] thoughts are parts that create our whole, right? And our whole is different if we don’t have all those parts… Collectively, we’re better than individually… Everyone has a contribution to make to this class.

As these quotes indicate, Ms. Skywalker sought to create safety by establishing *multiple perspectives* as part of the ethos of her classroom. When this norm was established, she believed, every student would be more willing to share because they knew that their unique perspective added something valuable to the class discussion. Rather than a classroom that compared different strategies in an evaluative way, Ms. Skywalker sought, through *multiple perspectives*, to establish a classroom that celebrated a diversity of mathematical viewpoints and considered such diversity as a strength. As was discussed in the micro strategies section, she overwhelmingly promoted *multiple perspectives* through direct prompts for additional solution strategies, representations, justifications, and computational strategies. However, she did make occasional normative comments to her class affirming *multiple perspectives*. The following statement, from October 2nd, was made to her students after they had worked in groups for some time on a
rich task. Notice the same recognition of cognitive diversity (rather than evaluative comparison) that Ms. Skywalker previously mentioned in interviews:

Ms. Skywalker: What I like about this rich task is that it honors each of your thinking as unique, right? I don’t expect you to all think about this problem the same way… I want each group to share their thinking… and I’m going to do this in a random order because no one’s thinking was necessarily “better” or “more advanced” than anyone else’s. They were just different, right? Whatever you needed to do to make sense of the problem.

By saying this, Ms. Skywalker emphasized to her students that everyone thinks differently and that this was okay (“I don’t expect you to all think about this problem the same way”). She downplayed evaluative comparisons between perspectives (“No one’s thinking was necessarily ‘better’ or ‘more advanced’ than anyone else’s”) and emphasized that it is more important to “make sense of the problem.” Multiple perspectives seemed to be the primary means that Ms. Skywalker employed to build a safe environment, but it was not the exclusive means. She also forbade students from waving their hands frantically and shouting out in response to her questions. In the following incident, on September 4th, several students had been excitedly saying, “I know why!” in response to a question. Ms. Skywalker used this behavior as a normative example to promote a safer and more considerate environment:

Ms. Skywalker: Hey! You calm down, “I know why.” Keep it in your head. How do you think that makes someone feel who doesn’t quite know why yet? Does that make them feel pretty good? No. So keep it in your head. We’re about respecting each other as learners here.

Normative examples such as this indicate that Ms. Skywalker expected her students to contribute towards creating a safe environment; it was not a unilateral effort on her part. Through interviews, Ms. Skywalker also informed me that she was especially gentle in correcting students’ errors during the initial weeks of the school year. Until the classroom had developed a sense of community, she recognized that students would be especially sensitive to potential embarrassment. Finally, the next two sub-strategies,
emphasizing understanding over performance and knowing individuals’ strengths and weaknesses, also contributed towards creating safety. These will be discussed in the upcoming paragraphs.

The third sub-strategy of creating a conducive classroom environment is emphasizing understanding over performance. A number of Ms. Skywalker’s students seemed to have a performance-oriented view of mathematics, a view that “doing math” is primarily about obtaining the correct answer as quickly as possible. Ms. Skywalker felt very strongly that this was an unhelpful mindset; she believed it encouraged rote memorization of facts and procedures devoid of mathematical understanding. She also believed it created the harmful and erroneous belief that being good at mathematics meant performing computations and solving problems quickly. In the interview quote below, Ms. Skywalker explained her views on this performance-oriented mentality:

Ms. Skywalker: So often in elementary school, being good at something is being fast. That’s such a battle for me sometimes… And sometimes my slower students are really great math reasoners. They think about things, but it just takes them a while… [The students] need to have [mental strategies for performing computations]. And they don’t have them, and then we just say, “More times tables, more times tables, more times tables!” That’s what happens. They do… times tables like every day in 4th grade… And I used to think that. “You just need to memorize these, you just need to memorize these, you just need to memorize these.” I don’t think they just need to memorize [their times tables] anymore. Why don’t they understand them?

In Ms. Skywalker’s mind, repeated drill and memorization did not necessarily engender understanding. Her mention of “that’s such a battle for me” refers to the fact that many of her performance-oriented students were resistant to changing their mindsets. As the following two interview quotes indicate, both high-ability and low-ability students demonstrated this performance-oriented mindset:

Ms. Skywalker: [Some of my students are] really bright math kids. [They’re] kind of use to being the go-to, “I’m the star of math.” That’s not what it’s about in my classroom.
Ms. Skywalker: [One of my low-ability kids] is a speed demon… My biggest challenge with him is, “Stop! Give yourself a chance to think about this.” He’s about, “Answer quick! Go, go, go!”… He’s really equating accomplishing things as finishing quickly. So I’ve really got to work with him on that.

Ms. Skywalker talked in great detail with me about her performance-oriented students, particularly her highly-capable ones. She believed that their resistance to changing mindsets stemmed from placing their identity and sense of self-worth in being faster than their peers. To combat the performance-oriented mentality, Ms. Skywalker employed a variety of strategies. She made numerous and emphatic normative comments that de-emphasized performance and stressed understanding. The following two quotes from her classroom exemplify the many similar normative comments that she made:

Arnold: The answer key fits the pattern.
Ms. Skywalker: Okay, I’m glad the answer key fits the pattern. I’m glad. That makes me happy. But the answer key doesn’t really explain to me any reasoning or thinking, does it? And that’s why our goal is not to match the answer key. We got that? Our goal is not to get the right answer. Our goal is to…
Arnold: Learn.
Ms. Skywalker: Learn and reason and think things out. Because you learn more when you explain stuff.

Ms. Skywalker: Math in the real world is not the right answer on demand! It’s can you communicate the answer, and how! How! Why! No one in your dad’s work is going to come in and go, “Square root of 34! Go!” That’s not how it works [in the real world]!… Have you ever felt slow in math? Guess what? It’s not a race. It’s not a race. It’s not about getting the right answer. It’s about thinking deeply. And reasoning, and thinking, and reasoning, and going back… “I’m smart so I get all the right answers.” No! Doesn’t work that way!

As these exemplars indicate, Ms. Skywalker repeatedly portrayed a performance-oriented view of math and an understanding-oriented view of math as at odds with each other. She stressed that a performance-oriented view of math was unhelpful for real-world use. Beyond her normative comments, she also rarely gave timed tests and refrained from calling on students who raised their hands first, believing that both of these things tacitly contributed to the performance view of math. Furthermore, Ms. Skywalker chose and
used tasks in a way that focused on concepts and ideas rather than speed. This will be explained in greater detail in the concept-oriented task philosophy section. Finally, this sub-strategy of emphasizing understanding over performance was closely related to the previous sub-component of creating a safe classroom environment. The norm of multiple perspectives, used to help create a safe environment, also helped to shift the focus of mathematical activity from the obtaining an answer to the process of reasoning. Conversely, deemphasizing performance helped to create a safe classroom environment by downplaying the competitive nature of the performance-oriented view of mathematics.

The fourth sub-strategy of creating a conducive classroom environment is the teacher knowing individual students’ strengths and weaknesses. This is perhaps best summarized by the following succinct interview comment:

Ms. Skywalker: You work with what you have.

Unlike the other conducive classroom environment sub-strategies, this particular sub-strategy was not apparent through observations alone. Because of its “behind-the-scenes” nature, Ms. Skywalker’s knowledge of her students only became apparent after it was explained during interviews. Ms. Skywalker made a concerted effort at the beginning of the academic year to know the strengths and weaknesses of each of her 26 students. She gave her students a questionnaire asking them about their learning styles and preferences as well as any information they wanted Ms. Skywalker to know about them. She also sent home questionnaires to students’ parents asking about their child’s strengths and weaknesses. She then compiled these results into a binder with detailed notes for each student and any particular accommodations that she planned to enact. She knew which students lacked self-confidence, were easily intimidated, and afraid of being wrong. She knew which students were performance-oriented and wanted (or were expected by their parents) to perform at the top of the class. This specialized knowledge guided her interactions throughout the upcoming months, influencing who she called on, how she corrected mistakes, how she dealt with misbehavior, and how she sequenced student work during times of sharing.
For example, on September 4th, Ms. Skywalker asked her students to visually represent the difference between odd and even numbers. Since she considered this to be a fairly straightforward and low-risk task, she purposefully asked many of her easily-intimidated students to share their representations with the class, thus helping to increase their mathematical confidence. Without interviews however, this would never have been apparent; observational data alone would have shown Ms. Skywalker asking a handful of students to share their visual representations with the class. The discreet role that Ms. Skywalker’s knowledge played in this particular example typifies the entire sub-strategy.

The fifth and final sub-strategy of creating a *conducive classroom environment* is the establishment of helpful logistics and procedures that assist learning. In Ms. Skywalker’s own words from an interview on September 11th:

Ms. Skywalker: One of [the foundation things to enable the Common Core math practices] is… an environment where… *there’s certain systems and procedures in place*...

What are these “systems and procedures” that Ms. Skywalker had in mind? Observational data revealed several different “systems” that Ms. Skywalker established in her classroom. Perhaps the most prominent was the student role of *business leader*. For almost every day that I observed, the class began mathematics by doing “morning work.” This work consisted of a simple worksheet or two that usually served as the launching point for mathematical discussions. The student who had the role of business leader would lead a whole-class review of the morning work, calling on their peers to answer various questions and provide justifications and different strategies. The role of business leader rotated through the entire class; every student served in this role multiple times throughout the school year. One of the main purposes of the business leader, was to prompt peers for justification if it was not provided. During the second week of school, on September 4th, Liz was serving as the business leader and forgot to prompt her peers to justify. Ms. Skywalker then stepped in with a normative explanation of the role of the business leader:

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19 For more detail on this “morning work,” refer to the *concept-oriented task philosophy* section later in this chapter.
Ms. Skywalker: So I’m helping Liz right now because this is her first week correcting morning work, but normally when [being the business leader] is your job, we ask for reasons why for answers. We ask them to explain their thinking… So often I would like you [when you answer] to start to say “[This is my answer] because…” So you give your answer and then you explain your thinking.

Thus, the “procedure” of reviewing morning work, led by the business leader, was intended by Ms. Skywalker to help establish the norm of justification. Another “procedure” that Ms. Skywalker often employed was “pair-share” or “think-pair-share.” During the “think” part of the sequence (if applicable), students were given time to silently consider a question or topic that had arisen in whole-class discourse. Next, during the “pair” part of the sequence, students would talk with a partner about the concept, claim, or problem. Finally, in the “share” part of the sequence, Ms. Skywalker would call the students back to whole-class discourse and allow for either a choral response or for the various pairs to share their thoughts with the rest of the class. The time allotted for entire sequence typically ranged from twenty seconds to several minutes. In an interview, Ms. Skywalker explained her rationale for frequently using pair-shares:

Ms. Skywalker: Engagement? Big deal for me. I think that’s something a lot of teachers miss, so kids miss stuff. That’s why [students] have holes [in their knowledge]. Because they’re not paying attention or whatever. So how are you making sure everyone’s engaged all of the time? I do a lot of think-pair-shares. I do a lot of “call out, we’ll all answer together.” Right? There’s a lot of little checks to make sure we’re all together.

Ms. Skywalker’s use of the pair-share sequence was motivated by her concern for student engagement, believing that it helped her students to better focus on the material. Thus, the pair-share sequence was another “procedure” that Ms. Skywalker employed to assist her students’ learning. However, implementing this procedure required a deliberate effort on Ms. Skywalker’s part. On several occasions during the beginning of the school year, Ms. Skywalker had to issue normative instruction regarding pair-shares. In the following incident on September 4th, Ms. Skywalker noticed that some of her students sat passively during a pair-share time and consequently issued the following correction to the entire class:
These sorts of incidents confirmed that establishing classroom procedures was not instantaneous, but rather required time and explicit instruction from Ms. Skywalker. Finally, the use of hand signals to indicate agreement or disagreement, seemed to be included as part of the “systems and procedures” Ms. Skywalker sought to establish. She frequently prompted and reminded her students to use hand signals to indicate agreement (or lack thereof) with an answer, idea, or strategy. Similar to the pair-share sequence, this “system” of hand signals supported learning by promoting student engagement.

Overall, the macro strategy of creating a conducive classroom environment passively supported the productive sociomathematical norms. By “passively supported,” I mean that this macro strategy did not directly promote the norms per se, but instead promoted general student engagement with mathematics. Or put another way, a conducive classroom environment worked to remove potential barriers that might prevent the emergence of the productive norms. By establishing helpful classroom procedures and making learning fun and exciting, Ms. Skywalker worked to increase student engagement and remove the barrier of student disengagement due to boredom. By creating a safe atmosphere, Ms. Skywalker encouraged students to actively participate in classroom discussion and removed the barrier of fear of humiliation. By emphasizing understanding over performance, Ms. Skywalker helped students to think about the process of mathematical reasoning and removed the barrier of a competitive mentality. Hence, in this manner, a conducive classroom environment laid a foundation for the productive norms to emerge.

**Foundational skills**

Ms. Skywalker seeks to teach her students foundational skills that allow them to engage in the productive sociomathematical norms. To do this, she promotes:

- **Active listening**
- **Use of visual representations**
- **Use of precise mathematical language**
Ms. Skywalker recognized that complex mathematical practices tacitly relied on more basic, “foundational” practices in order to function. She saw these foundational practices as necessary pre-requisites in order for the more complex practices to be successfully realized. In the following interview quote, Ms. Skywalker discusses Common Core math practice 3, “Construct viable arguments and critique the reasoning of others.” Note how she describes necessary foundational practices:

Ms. Skywalker: “Critique the reasoning of others.” I think I’m stepping there. Listening to others is the first step. Now, comparing your thinking to someone else’s, or your rationale to someone else’s, that might be this next step. Peter: And they have to get their reasoning out there so they can start critiquing it. Ms. Skywalker: Exactly! So we have first steps first. We can’t get there yet. There’s several things we need to do first. But I think we’re building towards that. And that’s the other thing that we were talking about: you can’t just come in do these [Common Core Math Practices]. I mean, you can understand them all you want, but there’s some foundation things that need to happen…You cannot have productive mathematical discourse… you can’t do any of those things without some foundation skills, if you will, by your kids. Can they listen to each other?

Ms. Skywalker’s view of complex math practices as built atop simpler, foundation practices can be visually represented by Figure 5-7 below.

Figure 5-7. A visual representation of Ms. Skywalker’s view of mathematical practices.

This leads naturally to the question of which foundational practices helped enable the productive norms in Ms. Skywalker’s classroom. Observational data revealed three practices that she taught her students in the initial months of the school year that helped support the productive norms. Each of these foundational practices will now be discussed in turn, as well as how that foundational practice enabled the productive norms.
The first foundational practice that Ms. Skywalker sought to promote was the norm of active listening. Although active listening was itself one of the five productive norms identified in phase one, it also supported the four sociomathematical norms. This supportive relationship was discussed in detail in Chapter 4. Furthermore, interviews with Ms. Skywalker revealed that she viewed active listening as a means towards establishing mathematical practices. In the following two interview transcripts, lines referring to active listening have been italicized. Note how Ms. Skywalker views listening skills in relation to mathematical practices:

Ms. Skywalker: “Critique the reasoning of others.” I think I’m stepping there. Listening to others is the first step. Now, comparing your thinking to someone else’s, or your rationale to someone else’s, that might be this next step.

Peter: And they have to get their reasoning out there so they can start critiquing it.

Ms. Skywalker: Exactly! So we have first steps first. We can’t get there yet. There’s several things we need to do first. But I think we’re building towards that. And that’s the other thing that we were talking about: you can’t just come in do these [Common Core Math Practices]. I mean, you can understand them all you want, but there’s some foundation things that need to happen…You cannot have productive mathematical discourse… you can’t do any of those things without some foundation skills, if you will, by your kids. Can they listen to each other?

Ms. Skywalker: I have found that it is helpful [to build students’ mathematical skills], and that’s why people come in and they’re like, “Oh my gosh, your kids are doing [amazing things]”… I have found more value in the share-out, in honoring their thinking. When [the students] see other people’s representations up there, they’re used to listening to other people, seeing their representations. So you get more from the [larger tasks] than I did when I wasn’t really developing those mental skills, mental practices ahead of time.

In the first transcript, Ms. Skywalker refers to listening skills as a “first step” towards establishing a more complex mathematical practice. Later in the same transcript, she mentions listening skills (“Can they listen to each other?”) as an example of a “foundation skill” necessary for productive mathematical discourse to occur. In the second transcript, Ms. Skywalker again mentions listening (“they’re used to listening to other people”) as an example of a “mental skill” or “mental practice” that helps enable more productive student mathematical activity. Therefore, Ms. Skywalker viewed active listening as a means to
establishing more mathematical goals rather than a goal in itself. This agreed with observational data, which indicated that *active listening* functioned in a supportive and enabling role for the sociomathematical norms. Thus, I found it reasonable to categorize *active listening* as part of Ms. Skywalker’s macro strategy of *foundation skills*.

The second foundational skill that Ms. Skywalker promoted was use of visual representations. She believed that the process of representing numbers, operations, and problem situations visually was a skill that needed to be purposefully taught to students. She explained this in the following interview transcript:

Ms. Skywalker: [The students] don’t know what the ways are to [visually] represent stuff, they’ve never done it in other grades. Right? They’re like, “What do you mean?”

Peter: Yeah, it’s kind of like a skill set that you develop.

Ms. Skywalker: It is!

Observational data showed that Ms. Skywalker made an intentional effort during the initial months of the school year to teach her students how to visually represent mathematical ideas. In some situations, she introduced visual representations herself through modeling, while in other situations, she directly prompted students to create them and then let individuals share their representations with the class.

Below, Figure 5-8 contains student-produced representations of even and odd, while Figure 5-9 contains representations of $3 \times 5$ that Ms. Skywalker introduced to the class:
Figures 5-8 and 5-9 are typical of the kinds of visual representations that the class drew early in the school year. In addition to representing parity and the operation of multiplication, the class also visually represented factors, primality, the commutative property of addition, fractions, decimals, and specific problem scenarios. All of these occurrences allowed students to develop the practice of visually representing various mathematical concepts. These visual representations then supported the four productive sociomathematical norms.
For example, students often justified claims by drawing and appealing to visual representations. To justify that an even number times an odd number results in an even number, students appealed to the visual representations shown below in Figure 5-10.

![Figure 5-10](image)

Figure 5-10. The class’s visual representations used to justify why an even times an odd results in an even number.

Note how these representations in Figure 5-10 build on both of the representations from Figures 5-8 and 5-9. To construct the representation above in Figure 5-10, students drew on the array model for multiplication initially introduced by Ms. Skywalker in Figure 5-9. They then “paired-up” the dots as they learned to do when Figure 5-8 was discussed. Together, this revealed the mathematical structure of the situation and allowed the students to understand why the result of an even times an odd was always an even number. Without these visual representations, it is questionable whether students would have been able to explain why this pattern occurred.

Use of visual representations also helped support the norm of computational strategies. Through their representations, students found new ways of conceptualizing computations. For example, when students were asked to perform 0.5/5, Janet used a visual representation shown below in Figure 5-11.
Janet used five colored erasers, letting each one represent 0.1. Together, she knew that they represented the quantity 0.5. She then explained how dividing by 5 meant splitting a quantity into 5 equal parts. For her collection of erasers, this meant that each part would consist of a single eraser, or 0.1. Hence, Janet used her erasers to visually represent and make sense out of the computation 0.5/5. This exemplar is typical of how students used visual representations to help conceptualize different computations.

Additionally, use of visual representations helped students to establish mathematical coherency. Reflecting on visual representations helped students to see how different mathematical concepts were related to each other. An exemplar of this is shown in Figure 5-12 below. On this occasion, the class had drawn four arrays that represented the number 15. Ms. Skywalker then listed the factors of 15 on the board next to the arrays. Through reflection, this allowed students to recognize that the number of factors a whole number has is equal to its number of arrays. Students then connected this to primality, recognizing that since a prime number has only two factors, it must have only two arrays. Hence, in this situation, the use of visual representations aided student reflection, helping students to generalize by relating different mathematical concepts.
Finally, use of visual representations supported the norm of *multiple perspectives*. The creation of different visual representations led to different perspectives on the same situation. For example, Janet’s use of erasers to represent $0.5/5$, shown earlier in Figure 5-11, offered a different way to consider that computation.

The third foundational skill that Ms. Skywalker promoted was the use of precise mathematical language. When talking with me about computational strategies, she mentioned the importance of providing students with necessary terminology so that they could verbalize their mathematical activity:

Ms. Skywalker: We used to have [computational] strategies on the board. Did you find a double? Did you do this? A landmark fraction? You’ve got to use all those strategies. Kids don’t even have labels for those things right now.

Lacking the relevant terminology, Ms. Skywalker believed students would experience greater difficulty in verbalizing their mathematical reasoning. She expressed her desire to be explicit with her students about mathematical terminology, helping them to connect their mathematical activity to these new terms:

Ms. Skywalker: I like the idea of being more explicit with kids. “You’re making a generalization. You’re justifying”… I want to try and do that a little bit more.
During class, Ms. Skywalker also spoke to her students about the importance of precision in language.

The following remark exemplifies the comments she made in this area:

Ms. Skywalker: Just like we work on proper grammar, there’s proper mathematical language and we want to make sure we’re being exact because it can make a pretty big difference.

Over the first two months of school year, Ms. Skywalker introduced or clarified a number of terms for her students including parentheses, order of operations, equivalent fractions, digit, place value, even, odd, area model, array, commutative property, associative property, distributive property, inverse, prime, composite, factor, exponents, and more. Generally, the term would first be defined, either by Ms. Skywalker, a student, or the dictionary. A brief discussion of the term would then typically follow in which examples of the term would be given and the term’s relation to other terms would be discussed. This type discussion often supported coherency. An exemplar of this occurred when the class discussed the expression $1 + (17 - 1)$. Ms. Skywalker took the opportunity to introduce the term parentheses:

Ms. Skywalker: There’s something in this equation that is not in all equations… do you see something in here that we don’t always see? … George, what do you see?
George: I forget what they’re called.
Ms. Skywalker: Valerie?
Valerie: Parentheses.

Ms. Skywalker: So there’s parentheses in here. Why are they in there?
Tim: The parentheses are there because that’s the equation you have to do first.

Ms. Skywalker then used this dialogue to segue to talking about the order of operations and why it was a useful and clarifying tool. Hence, in this episode, a relevant term was introduced (parentheses) and its relation to another term (order of operations) was discussed, thereby helping students to build a coherent understanding of mathematics. The new term’s meaning and usefulness were also explained. Other terminological introductions followed a roughly similar pattern as this.
Use of precise terminology facilitated discussion of *multiple perspectives*, particularly when these perspectives were dealt with *computational strategies*. As Ms. Skywalker mentioned earlier, terms such as “doubling” and “landmarks” gave labels to students’ mathematical activity that might have otherwise gone unlabeled. On one occasion, the class even invented a term to describe a particular computational strategy. Carlos had used addition to perform the operation of subtraction by “adding up” to find the difference between the two numbers in question. Ms. Skywalker decided to label the use of addition to perform subtraction as “The Carlos Strategy.” Although this is clearly not an official mathematical term, it was nevertheless a *precise* term because the class understood what it specifically referred to. By coining the term “The Carlos Strategy,” Ms. Skywalker gave her students a concise label to refer to the use of addition to perform subtraction, thereby simplifying their discussion of this particular strategy.

Precise terminology also helped students to make justifications in certain situations. By clearly understanding the definition of relevant mathematical terms, students could then appeal to these definitions as part of a basis in justifying their claims. An exemplar of this occurred as Rebecca attempted to justify that the number 15 was composite. Notice how terminology quickly became the limiting factor in this situation:

Ms. Skywalker: Is 15 prime or composite?
Rebecca: Composite because…
Ms. Skywalker: It’s composite? Why? What do you think? Before you start, do you remember what a composite number is? Do you know what a prime number is?
Rebecca: I think so.
Ms. Skywalker: What do you think a prime number is?
Rebecca: Um, I think a prime number is like 5 and 3, but I don’t know what it is.

Because Rebecca lacked a clear understanding of the terms “prime” and “composite,” she was unable to justify her claim that 15 was composite. After this dialogue, Ms. Skywalker had the class open their math books and read the definitions of prime and composite. Once they understood what these terms meant, students were able to justify that 15 was in fact composite by appealing to the definition of composite. In
this manner, a clear understanding of relevant terminology aided students in successfully making justifications.

Concept-oriented task philosophy

Ms. Skywalker uses mathematical tasks in a way that:

- Prioritizes concepts over task completion
- Builds desired mathematical skills and practices in her students over time

Ms. Skywalker’s well thought-out task philosophy was a key macro strategy. Her overriding goal with mathematical tasks was for her students to learn concepts, rather than simply completing the task. Consequently, Ms. Skywalker demonstrated great flexibility in her lesson implementation. She frequently deviated from the planned lesson in order to pursue an idea or topic that had arisen in class discourse, a decision that she and her students referred to as “going off on a tangent.” Ms. Skywalker felt that these tangents were important because they represented opportunities to explore relevant ideas that currently had the students’ attention. She explains these ideas in the following two interview quotes:

Ms. Skywalker: I joke about it a lot: “Oh, Ms. Skywalker’s going off on a tangent,” or, “Oh, we’re going off on a tangent.” But I hope that what you’ll see and what [the students] will start to see is: we’re taking advantage of a learning opportunity. And we’re going off. It’s not like it’s not related to anything.

Ms. Skywalker: I go off on tangents all the time, but [my students] are learning. It’s really important work. I think it [should be] called, taking advantage of a learning opportunity, and I feel that that’s a teachable moment… And you have to take advantage of that.

These quotes reveal that Ms. Skywalker thinks of lessons in terms of their constituent concepts. She believes that a prime learning opportunity has arisen when a concept has come to the forefront of discourse. At that particular moment, the concept is more likely to hold students’ attention and interest than at a later time. This interest and relevance can be leveraged by pursuing the concept and “going off on a tangent.” As a result of pursuing tangents, Ms. Skywalker frequently was unable to finish her
original planned lesson. However, she felt that this was an acceptable trade-off for the learning that occurred during the tangent:

Ms. Skywalker: Gosh! I always run out of time, but it’s okay because my kids are learning good stuff.
Peter: It seems like you’re willing to take the tangents if you think it will result in learning.
Ms. Skywalker: That’s the point! I will take tangents, but it’s not like I take them for no good reason… It’s so powerful for [the students] to make these connections…. I think [taking tangents] is good teaching practice… I do.

As Ms. Skywalker implies in this quote (“I think [taking tangents] is good teaching practice”), she felt strongly about her practice of taking tangents. In fact, she felt so strongly about it that she did not even like the term “tangents” because she felt that it devalued the learning that they produced:

Ms. Skywalker: I don’t like saying, “Going off on tangents,” or as [other teachers] call them “bird-walks.” That’s kind of an education term. And the reason I don’t like it is that it devalues it. It devalues it, doesn’t it?
Peter: It makes it sound like, “Alright, let’s get back to the real business now.”
Ms. Skywalker: Exactly! And half of the stuff that [my students are going to teach to another 5th grade class], they got in tangents. You know what I mean: where we went off a little bit… I have ten year olds saying stuff that I don’t know if high school kids could voice in some ways, right?

In Ms. Skywalker’s mind, tangents were not a distraction or sideshow from the “real learning.” Tangents were an essential part of the real learning. She was emphatic during interviews that the tangents she took were not random or disorganized, but rather were intentional, purposeful, and beneficial for her students.

Ms. Skywalker’s tangents were indicative of her flexibility in teaching. She strongly believed that flexibility was an essential quality of effective teaching. Although Ms. Skywalker recognized that lesson planning and organization were important, she saw a potential danger in being too organized and rigid, as the following two interview quotes reveal:

Ms. Skywalker: You can plan out something… but you work with what you have. And different kids have different understandings, so the “go with the plan” may not work for all kids.
Ms. Skywalker: I think of this one teacher that teaches [at my colleague’s school], and she’s so, “This assessment goes with this, and this goes with this, and this goes with this, and this goes with this, and this is what’s next, and this is what we’re supposed to do, and these are the worksheets that go with it.” She’s very thorough, she’s very complete, she’s very aligned. These are all good things. But not very teachable-moment [oriented]. Not very, “Hey, let’s try this!”

Ms. Skywalker recognized that thoroughness and organization in planning were beneficial qualities. However, when emphasized too strongly, she saw these qualities as becoming counterproductive to the goal of student learning. When a teacher was overly focused on following a prescribed plan and completing a lesson, Ms. Skywalker believed that valuable teachable moments would be overlooked. She also saw the teacher’s role as responsive and adaptive to the students’ current level of understanding. Therefore, in her mind, lesson plans were somewhat conditional and subject to change as needed.

Tangents were also representative of Ms. Skywalker’s general task philosophy of “depth over breadth.” In other words, she believed that deeply exploring a single, smaller, simpler task was better than tackling multiple tasks or larger tasks on a more superficial level. What was important to her was not the complexity of the task itself or the impressiveness of it, but the quality of mathematical reasoning that students were actually engaging in. Ms. Skywalker even believed that large, complex tasks could potentially be counterproductive because they tended to draw the teacher’s focus away from student reasoning to the task itself and the teacher’s role in it. She explains these views in the following interview quote:

Ms. Skywalker: The other thing that can be overwhelming... is: Is this task good enough? Is this a rich enough task? ... I remember when I did National Board... and you try to get these amazing tasks, and my advisor... kept saying, “This is a common problem most people make. They think they have to have these amazing tasks.” You know, “My students were basically designing their own government!” ... When you focus on a task, it’s almost like you’re focusing on yourself as a teacher. “I can design the best task.” It’s about the kids. And so if I remind myself, “What are they doing?” Not, “What am I doing?” What are they doing? ... It is about me, I know I design the task, and there should be some thought into it. I’m not saying that’s not important. It’s the idea that if there’s mathematical reasoning going on, if kids are engaged and they see the utility of math, and they can make connections, isn’t that more
powerful than, “I did [a big task] without [all these qualities]”? To me, it seems like they go together. I’m saying [big tasks] are very important…. It is about the tasks, don’t get me wrong. But it’s also about… the practices, the Common Core Practices for mathematical thinking, the 5 Practices, all of those things.

In this interview quote, Ms. Skywalker reiterated her views that her focus during tasks is on her students, their reasoning and the mathematical practices that they are engaging in. The task itself, while certainly important, should not eclipse these considerations. Ms. Skywalker went on to explain that student reasoning and mathematical practices are independent of task size and task complexity:

Ms. Skywalker:  I think [teachers] think that, “Oh, I don’t have time for rich task so I’m not going to get justifications.” Not the case! I actually do very few rich tasks… but we’ve been justifying like crazy.

Ms. Skywalker:  I don’t have to go out and find richer tasks! We can make almost any task rich.

In other words, Ms. Skywalker did not see student reasoning and mathematical practices as dependent on large, open-ended tasks. By remaining concept-focused and pursuing relevant tangents, Ms. Skywalker could elicit mathematical reasoning through even simple problems.

This concept-oriented task philosophy supported the productive sociomathematical norms by creating numerous opportunities for students to practice the norms. For example, many of the tasks Ms. Skywalker selected and the tangents she pursued focused on creating coherency through making connections. For example, on September 4th, the class was looking at the equation $50 - n = 30$ and trying to identify the value of $n$. One student mentioned how he had added up from 30 to get 50, and discovered a difference of 20. Ms. Skywalker then decided to investigate why addition could be used to solve a subtraction problem. This resulted in a 7 minute tangent in which the class discussed the concept of inverse, and also identified other inverse operations. Through this tangent, the class recognized that the particular solution strategy of “adding up” was merely one possible manifestation of the broader concept of inverses. Hence in this situation, Ms. Skywalker’s task philosophy provided an opportunity for students...
to establish *coherency*. Some of the other mathematical activities were evidently selected by Ms. Skywalker specifically for the opportunity they afforded students to establish *coherency*. For example, on September 11th, Ms. Skywalker wrote different powers of ten on the board, shown below in Figure 5-13. As she wrote, she asked students to anticipate what the next line would contain, thus encouraging them to identify the structural features of the situation and generalize. Therefore, this activity was specifically designed to allow opportunities for *coherency*.

![Figure 5-13. The powers of ten that Ms. Skywalker wrote on the board.](image)

Ms. Skywalker’s *concept-oriented task philosophy* also provided frequent opportunities for justifications to occur. When possible, she identified claims that students had made, whether implicitly or explicitly, and pressed for justification. For example, on September 8th, a student answered “1/2 or 2/4 or 4/8” to a relatively simple problem of expressing what fraction of a pie was shaded. Ms. Skywalker recognized the implicit claim that these fractions represented the same quantity. She explicated this to her class and then pressed for a justification of why they were the same quantity. This prompted a 13 minute tangent about the concept of equivalent fractions and multiple justifications of why certain fractions were equivalent. Although the original task question did not mention equivalent fractions, Ms. Skywalker’s
task philosophy led her to draw out this concept, thereby providing an opportunity to justify in the process. This example is representative of many of the justifications that were elicited through tangents. Other tasks were selected by Ms. Skywalker specifically for their justification potential. For example, on multiple occasions, the class did a worksheet called Today’s Number. This worksheet involved selecting a whole number and then justifying specific properties of it (e.g. parity and primality). Ms. Skywalker explained to me during interviews that she liked this activity in part because it allowed students to gain practice justifying.

In a similar manner, the norm of computational strategies was also promoted through Ms. Skywalker’s task philosophy. Many of her tangents were solely focused on exploring different ways of performing a computation. For example, on five different days, the computations $180 \div 26$, $0.73 - 0.32$, $4.3 \times 3$, $36 \div 100$, and $360 - 90$ respectively were discussed in considerable detail, with many different strategies and ways of conceptualizing the computation being discussed. Additionally, some of Ms. Skywalker’s planned mathematical tasks were primarily focused on promoting different computational strategies. For example, she had her students spend several days investigating non-standard multiplication algorithms.

Finally, the concept-oriented task philosophy also supported the norm of multiple perspectives. Because of her belief in depth over breadth of tasks, Ms. Skywalker was willing to regularly prompt for additional strategies, answers, or representations, and take the necessary time to discuss them.

However, Ms. Skywalker’s task philosophy went beyond simply focusing on depth over breadth and pursuing relevant tangents. It also extended longitudinally. That is, she had a clear idea of how she wanted her tasks to progress and build on each other over time. Ms. Skywalker began the year with shorter, simpler tasks designed to build foundation skills in her students. Through these simpler tasks, she systematically introduced skills and ideas to her students that would be built upon and further developed later in the school year. In the following interview quote, she explains this idea of starting with short, simple tasks:
Ms. Skywalker: This idea of systematic, explicit instruction… that’s something that really stood out to me [during my teacher preparation program]. This idea that it’s not just, “Hey, what do we got! Throw stuff at kids!” This idea of systematically, explicitly introducing ideas to them… These tasks are engaging, if they have access to them. So how do give them access?… You start with 100s chart at the beginning of the year. What patterns do you see? Why do you think those are happening? And I think that what happens is, some of us as teachers, and I thought this: “That’s obvious. That’s basic, everyone knows why this is.” Well actually, they don’t. Actually, kids have very seldom said, [“That’s obvious.”] I use even and odd number tasks at the beginning of the year. I used tasks that people would be like, “What are you doing in fifth grade?”

Ms. Skywalker admitted that many of her tasks used at the beginning of the year might seem too “basic” for the 5th grade level. However, she had specific goals that she intended to accomplish with them. Her first mathematical task of the school year was the 100s chart. For this task, students worked in small groups to write the numbers 1 through 100 in an array with dimensions of their choosing. Students were then supposed to identify patterns occurring within the chart and justify why those patterns were happening. This seemingly-simple task accomplished a number of things. First, it allowed students to focus on the practices of generalizing and justifying without the added difficulty of new, unfamiliar mathematical content. Secondly, it emphasized understanding and deemphasized performance; there was no definite “answer” that students could race to obtain. Rather, the goal was to identify patterns and understand why they were happening. Finally, the task was fun for students. As the first math task of the year, it began shaping their expectations for what mathematics in Ms. Skywalker’s class would be like.

Ms. Skywalker explains these aspects of the 100s chart in the interview quote below:

Peter: So would it be fair to say that when you teach your kids how to reason and make connections, you start with content that’s comfortably within their grasp?

Ms. Skywalker: 100s chart. They’ve seen 100s chart a hundred times. And you should see how jacked up they get—you’ve seen them! How jacked up do they get about finding patterns in the 100s chart?

Peter: They get really excited.

Ms. Skywalker: What’s wrong with that? “Oh this is a pattern. Wait, there’s a reason why this pattern is working? It’s not just because my teacher told me or because I learned that in first grade?” Do you hear kids saying, “I learned that in first grade”? No!
After the initial 100s chart task, Ms. Skywalker continued using relatively “basic” tasks for the first month of the school year. However, even these basic tasks built upon each other. For example, on September 4th, Ms. Skywalker had her students visually represent even and odd. She purposefully had many of the less-confident students share their representations with the class since she saw this as a “low-risk” activity and hence, an opportunity for these students to gain mathematical confidence. On the same day, she also introduced the array model for multiplication. Nine days later, on September 15th, Ms. Skywalker introduced the Today’s Number worksheet for the first time. For this worksheet, the class would pick a whole number and then justify whether it was even or odd and prime or composite. They would then lists its factors and arrays, and create ten different expressions that represented that number. In justifying parity, students utilized the visual representations that the class had discussed on September 4th. On September 15th, the class also justified additive and multiplicative parity properties. Later in the fall semester, the class did a larger, open-ended, rich task involving the difference of consecutive square numbers. The justification elicited by this task involved visual representations of parity and the additive parity properties. Hence, these tasks exemplify Ms. Skywalker’s “systematic, explicit instruction.” They systematically built upon each other, taking ideas previously introduced and using or extending them. Ms. Skywalker’s goal in using smaller tasks during the initial month of the school year was to eventually work towards doing larger, more complex, rich tasks. However, she felt that students needed certain mathematical practices in order for these rich tasks to be productive.
Figure 5-14. Visual representations from various tasks throughout the fall semester that exemplify Ms. Skywalker’s systematic instruction.
She explains her task progression in the following interview quote from September 18th:

Ms. Skywalker: I won’t always be teaching math this way. You know what I mean? I always will reinforce certain things. That’s just the way I am. But we haven’t gotten into a lot of rich tasks yet, and that’s where I’m hoping to get the transfer from this work. Right?

Peter: So you start with these kinds of, maybe, simpler, shorter tasks?

Ms. Skywalker: Yes, towards the beginning of the year. And then we’re going to start doing rich tasks. But right now, we’re doing some, I like to call them skill-building kinds of things, like Today’s Number, like Strings…

…

Peter: So then, when you give them a rich task, they’re not just like, “What do we do? We don’t know.” They already have this idea of: look for why, look for why.

Ms. Skywalker: Yes! And it’s great! I have found that it is helpful, and that’s why people come in and they’re like, “Oh my gosh, your kids are doing [amazing things]”… I have found more value in the share-out, in honoring their thinking. When they see other people’s representations up there, they’re used to listening to other people, seeing their representations. So you get more from the rich task than I did when I wasn’t really developing those mental skills, mental practices ahead of time.

The first rich task of the school year was done on October 2nd. From that point on, Ms. Skywalker would periodically use these sorts of larger, less-structured tasks. After October 2nd, she also stopped using the Today’s Number worksheet. She later explained to me how it “had served its purpose” in teaching students to justify, create multiple representations of the same number, and recognize connections between factors, arrays, parity, and primality.

This longitudinal aspect of Ms. Skywalker’s task philosophy also supported the establishment of the productive sociomathematical norms, particularly justification and coherency. By beginning with mathematical content that was comfortable and familiar to students (e.g. 100s charts and parity), Ms. Skywalker allowed them to develop the practices of justifying, making connections, and thinking about concepts from multiple perspectives.
A model to represent the macro strategies

In light of the macro strategies just discussed and the interactions between them, I put forth the following model, Figure 5-15, as a way to visually represent the macro strategies Ms. Skywalker employed to support the establishment of the productive sociomathematical norms.

![Figure 5-15](image)

Figure 5-15. A visual representation of the macro strategies Ms. Skywalker employed.

The three ovals in the figure represent the three macro strategies that have just been discussed, while the box in the center represents the four productive sociomathematical norms identified in Phase One of the study. The arrows that point from each oval towards the central box represent the fact that each of the macro strategies helped to support the four productive sociomathematical norms. How exactly each macro strategy supported the productive sociomathematical norms was elaborated upon earlier during the discussion of the macro strategies in the preceding sections. Notice also in Figure 5-15 that there is a bidirectional arrow between each pair of ovals. This signifies that each macro strategy also helped to buttress the other two. This interplay between each of the three macro strategies will now be systematically discussed.

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20 Recall that active listening is included within foundational skills.
Recall that the macro strategy of a *conducive classroom environment* meant creating an environment where learning was exciting, fun, and safe, where understanding was emphasized over performance, where helpful logistics and procedures were established, and where students were known by their teacher. Through doing this, Ms. Skywalker hoped to increase overall student participation in mathematical activity and discourse and to place the class’s focus on the process of mathematical reasoning rather than obtaining an answer. Greater participation in mathematical discourse allowed students more opportunities to practice active listening and use precise mathematical terminology. Therefore, a *conducive classroom environment* passively supported *foundational skills* by allowing more frequent opportunities for students to practice their foundational skills. An emphasis on the process of reasoning, rather than performance, helped to support Ms. Skywalker’s task philosophy of depth over breadth. Rather than racing to finish tasks before their peers, students were more likely to consider the concepts inherent in each task. Therefore, a *conducive classroom environment* also helped support the *concept-oriented task philosophy*.

Recall that the macro strategy of *student foundational skills* consisted of the skills of active listening, using visual representations, and using precise mathematical terminology. The skill of active listening helped to support some of the procedures Ms. Skywalker had established in her classroom, particularly the pair-share procedure and the business leader procedure\(^{21}\) because the effectiveness of these two procedures depended upon students listening to each other. The use of visual representations helped to emphasize conceptual understanding over against a performance-oriented view of mathematics by focusing students’ attention on the mathematical structure inherent in the problem. In these ways, *foundational skills* helped to buttress a *conducive classroom environment*. Use of visual representations frequently helped students explore a mathematical task in greater detail. For example, when investigating multiplicative parity properties, the class used the visual representations shown below in Figure 5-16. This allowed them to see the mathematical structure of the situation and justify why these properties were occurring. Without the visual representations, it is questionable whether the students would have been

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\(^{21}\) A description of the pair-share and business leader procedures was given earlier in this chapter.
able to recognize this structure and produce a justification. Thus, in situations such as this, foundational skills also helped to support a concept-oriented task philosophy.

Figure 5-16. The class’s visual representations used to justify why an even times an odd results in an even number.

Recall that Ms. Skywalker’s concept-oriented task philosophy meant focusing on the mathematical concepts inherent in tasks, rather than simply focusing on completing numerous tasks or doing “impressive” tasks. It also meant scaffolding tasks over the first month of the school year to develop skills and practices. This scaffolding of tasks was intended to promote, among other things, use of visual representations. For example, on September 4th, part of Ms. Skywalker’s task was for students to visually represent even and odd, shown below in Figure 5-17. Additionally, part of the Today’s Number worksheet involved students drawing arrays to represent the chosen number, shown below in Figure 5-18.

Figure 5-17. Visual representations of even and odd that students created.
Other tasks helped to motivate the need for precise terminology. For example, on October 20th, one of the tangents that the class pursued stemmed from a problem asking students to draw a polygon with four right angles. Janet drew a polygon with three right interior angles and one right exterior angle, shown below in Figure 5-19. This led to disagreement among the students and ultimately motivated the terms “interior angle” and “exterior angle.” Several other similar incidents occurred where the task naturally led to the introduction of new, more precise terms.
Hence, the *concept-oriented task philosophy* helped support *foundational skills*. Ms. Skywalker’s task philosophy also led to her selecting tasks that were fun for her students. For example, the 100s chart mentioned earlier was enjoyable for students because of its exploratory nature in finding patterns. Recall Ms. Skywalker’s interview comments on how her students enjoyed this task:

**Ms. Skywalker:** They’ve seen 100s chart a hundred times. And you should see how jacked up they get—you’ve seen them! How jacked up do they get about finding patterns in the 100s chart?

**Peter:** They get really excited.

**Ms. Skywalker:** What’s wrong with that? “Oh this is a pattern. Wait, there’s a reason why this pattern is working? It’s not just because my teacher told me or because I learned that in first grade?” Do you hear kids saying, “I learned that in first grade”? No!

The conceptual focus of the task philosophy also worked to deemphasize performance and promote understanding. For example, one of Ms. Skywalker’s tasks was for students to investigate non-standard multiplication algorithms and explain how and why they worked. Since students already knew the standard multiplication algorithm, the focus of this task was not on obtaining a numerical answer. Rather, the focus was on developing understanding. Therefore, this task naturally emphasized understanding over against performance. Hence, the *concept-oriented task philosophy* also helped support a *conducive classroom environment*.

What Figure 5-19 and the preceding discussion indicate is that Ms. Skywalker’s three macro strategies were inextricably linked. They mutually supported each other in a synergistic fashion as they removed barriers to and provided opportunities for the successful emergence of the productive sociomathematical norms.

**Teacher strategies in light of the research questions**

This chapter has presented findings from phase two of the study. Phase two was intended to address the second research question:
What strategies can a teacher use to establish these norms [identified in phase one] in their classroom?

Five norms were identified in phase one: active listening, coherency, justification, computational strategies, and multiple perspectives. It was discovered that Ms. Skywalker used a combination of both narrowly-focused strategies with a more direct, immediate effect (micro strategies) as well as broader strategies that focused on the class’s long-term development and exerted a more indirect effect (macro strategies). Ms. Skywalker used four micro strategies to establish the phase one norms: direct prompts, normative comments, highlighting positive examples, and modeling. The relative frequency of these micro strategies was given in Figure 5-1 and varied depending on the particular norm in question. Ms. Skywalker also used three macro strategies to support development of the sociomathematical norms: creating a conducive classroom environment, teaching students foundational skills, and employing a concept-oriented task philosophy. These macro strategies synergistically supported each other and helped encourage development of the sociomathematical norms by removing barriers to the norms’ emergence, equipping students with necessary skills to participate in the norms, and providing opportunities for the norms to be practiced.
Chapter 6: Discussion, Implications, and Limitations of the Results

The intent of this study was to answer the following two research questions:

1. What social and sociomathematical norms are associated with mathematically productive discourse?

2. What strategies can a teacher use to establish these norms in their classroom?

Chapter 4 addressed the results pertaining to the first research question while chapter 5 addressed results pertaining to the second. This final chapter now presents a discussion of the results. I will first briefly summarize the results from both chapters 4 and 5. Next, I will address three questions that naturally arise from an initial consideration of the results: How dependent were these results upon Ms. Skywalker herself and her particular group of students from the 2014–2015 school year, what is the relationship between the macro and micro strategies, and how applicable are these results to other grade levels? After this, I will present some potential implications of the study, interacting with other research literature in the process. Finally, I will consider several methodological limitations of the study.

A brief summary of the results

Five norms were identified in Ms. Skywalker’s classroom. One of these, active listening, was a social norm. The other four, coherency, justification, computational strategies, and multiple perspectives were sociomathematical norms. These four sociomathematical norms were presented in Figure 4-8 within a Venn diagram to indicate their interrelated nature. To establish these norms, Ms. Skywalker employed both micro and macro strategies. Micro strategies were narrowly-focused and included teacher actions in light of their more immediate effect. Four micro strategies were identified: direct prompts, normative comments, highlighting a positive example, and modeling. By contrast, macro strategies were broadly-focused and included teacher actions in light of their long-term goals. Three macro strategies were
identified: creating a conducive classroom atmosphere, establishing foundational skills, and maintaining a concept-oriented task philosophy.

**The dependence of the results on this particular case**

Perhaps the most prominent question that naturally arises when considering the results is the extent to which they are dependent upon this particular case. In other words, what conclusions can be drawn from this study and what cautions must be observed when doing so? For clarification, it must be emphasized that the goal of this study is not to produce a generalizable theory of norms or norm development. However, it remains a pertinent question to consider if similar results would even be possible with a different teacher and a different group of students. Is there some unique quality of either Ms. Skywalker or her 2014–2015 class without which these results would be impossible? In addressing this question, I will first consider Ms. Skywalker herself and then her students. In discussing Ms. Skywalker, I will refer to Hill, Ball, and Schilling’s (2008) well-known model delineating the various components of mathematical knowledge required for teaching. Their model first decomposes this knowledge into two broad areas: knowledge of mathematics *itself*, and knowledge of mathematics *pedagogy*. Both of these components are then, in turn, further decomposed into more specific areas. For more information on the model, the reader should refer to Hill, Ball, and Schilling (2008).

Perhaps the most obvious and unique qualification of Ms. Skywalker was her participation in the *Making Mathematical Reasoning Explicit (MMRE)* project. *MMRE* provided extensive professional development to its teachers over a 3-year period and was specifically focused on both deepening their content knowledge and increasing their ability to elicit student-voiced generalizations and justifications. During the 2014–2015 school year, when my observations were conducted, Ms. Skywalker was in her final year of professional development with *MMRE*. This raises the question: to what extent were the norms and teacher strategies identified in this study dependent on Ms. Skywalker’s experience with *MMRE*? If Ms. Skywalker had *not* participated in *MMRE* professional development, would she still have been able to utilize the same strategies to successfully establish the same norms? To address this question,
one must first know how the *MMRE* had influenced Ms. Skywalker. She had plenty to say about her professional development experience during interviews. One of the largest benefits she recognized from the *MMRE* program was her increased confidence in teaching mathematics, as she explained in the quote below:

Ms. Skywalker: I was really intimidated by, when I taught math before [*MMRE*], and I think this is something that so many elementary teachers have: I didn’t think I knew enough math.

She went on to explain how these feelings of intimidation had previously affected her teaching:

Ms. Skywalker: Eight years ago, if I had a [highly-capable] student that shows me something… and I don’t get it, and it’s not the standard, accepted, best way to do it, I would have been like, “Okay, that’s neat. Next!” I wouldn’t have tried to [ask], “What were you thinking? Why did you do that? Can I connect it to something somebody else thought? Do you guys see this?”

Peter: So does that make it, when you feel like, “Well maybe I don’t know math well enough,” as a teacher, does that make you hesitant to probe for student reasoning?

Ms. Skywalker: Yes. Yes it does. I may not understand the reasoning. And I think teachers do that all the time… [Teachers] are really fearful of basically showing that they’re stupid. They’re just like our students! They don’t want people to think that they’re dumb.

The impact of this intimidation was not strictly psychological; it made Ms. Skywalker less likely to elicit and probe for student reasoning. In other words, this intimidation made her less likely to employ the micro strategy of direct prompts. With less prompting and investigation of student reasoning, it is less likely that Ms. Skywalker could have successfully established the four sociomathematical norms.

Evidently then, *MMRE* increased Ms. Skywalker’s confidence in teaching mathematics. But what specifically about *MMRE* helped to increase her confidence? She credited the idea of *multiple perspectives* as a major source of confidence, explaining why in the interview quote below:
Ms. Skywalker: I’d look at [a math problem] and I’d see a [solution] path. I was one of those kids that once I thought I had a solution, I couldn’t look at alternate things… I couldn’t back myself up and go, “Are there multiple representations of this?” That idea of multiple representations blew my mind. I never knew there were multiple representations… I never had that idea until MMRE… I do think one of the things MMRE does is it made me feel more confident as a math teacher…. Sometimes it isn’t even that you guys taught me more math… Yes, I learned more math, don’t get me wrong. And you increased my confidence. But more than that… It’s not even just increasing my capacity to do math. By increasing my capacity to consider other things, that has increased my capacity to do math. Does that make sense? I see someone else’s representation and I’m like, “Huh! So if that’s related to that, and that’s related to that… Ohh! Wow! That’s what base-ten means.”

In this quote, Ms. Skywalker referred to the idea of multiple perspectives as the “capacity to consider other things,” contrasting this with the inability to “look at alternate things.” The context of the quote indicates that by “things,” Ms. Skywalker means other solution paths and representations of the same problem. She seems to view multiple perspectives as a sort of skill that she learned during the MMRE program. This skill has allowed her to successfully interpret and learn from different perspectives that her students put forth. Through embracing multiple perspectives, she has been able to deepen her own mathematical content knowledge through her students’ work. She explains this in the following interview quote:

Peter: So your understanding grows from maybe even from things your students are doing and ways they’re bringing in connections?
Ms. Skywalker: Yes! … I tell my kids quite often, “I learned something new today. I hadn’t thought about that.” That idea of being a continual learner was reinforced through MMRE. That we’re never done.

Hence, personally learning the skill of multiple perspectives allowed Ms. Skywalker to go from feeling intimidated by unfamiliar perspectives to using them to deepen her own mathematical content knowledge. Ms. Skywalker’s personal adoption of multiple perspectives was a crucial skill undergirding her confidence in eliciting student mathematical reasoning. In addition to this new skill, Ms. Skywalker’s teaching confidence was also bolstered by the increased mathematical content knowledge that MMRE had imparted. When she referred to her increased content knowledge, Ms. Skywalker was most likely
referring to specialized content knowledge (SCK). SCK is the knowledge that teachers draw on to create and adjust the difficulty of examples, represent mathematical ideas, make sense of unusual student explanations, explain rules and procedures, and ask mathematical questions (Hill, et al., 2008). This knowledge itself is not usually taught to students, but rather functions in a behind-the-scenes manner within the teacher’s mind, enabling them to quickly and flexibly adapt to changing mathematical circumstances.

Besides augmenting her mathematical confidence, Ms. Skywalker’s enhanced SCK also enabled many of the strategies that she employed. For example, she taught her students the difference between a lower-level justification (e.g. appealing to an authority figure) and a higher-level justification (i.e. appealing to mathematical reasons). Through normative comments and direct prompts, she continued to reinforce her expectation for higher-level justifications. However, in order to do any of this, she had to first personally understand the difference between a lower-level and a higher-level justification and recognize when they were present in classroom discourse. Besides justification, Ms. Skywalker’s SCK also undergirded her micro strategies for the norm of coherency. In order to directly model mathematical coherency, Ms. Skywalker had to first establish her own. Likewise, before prompting her students to recognize a mathematical connection, she had to first know that such a connection existed. During interviews, Ms. Skywalker directly mentioned the influence of content knowledge on her ability to establish mathematical coherency:

Ms. Skywalker: There are places in mathematics that I would not look to make connections before [MMRE]. I wouldn’t. And now I do because of the two-fold purpose of MMRE. And that is to increase my math knowledge… and to help facilitate that in my students. So because of that, I look for things that I hadn’t looked for before.

This statement agrees with the relationship that researchers have noted between teacher content knowledge and teaching practice. Cohen (1990) observed a second-grade teacher, Mrs. Oublier, whose limited SCK prevented her from pressing students for conceptual understanding, even when there were frequent opportunities to do so. He concluded that Mrs. Oublier’s “relatively superficial knowledge of
[mathematics] insulated her from even a glimpse of many things she might have done to deepen students’ understanding” (p. 322). The presence, or absence, of content knowledge has also been found to impact a teacher’s task philosophy in a similar manner. Cohen (1990) noticed that Mrs. Oublier used many tasks that had potential for exploring and illuminating mathematical concepts. However, this potential remained unrealized due to lack of teacher content knowledge. Simply put, Mrs. Oublier did not recognize the key mathematical concepts in her tasks and consequently did not focus students’ attention on them. By contrast, Ms. Skywalker’s well-developed SCK enabled her macro strategy of a *concept-oriented task philosophy*. In order to remain concept-focused when using tasks, Ms. Skywalker had to first recognize and understand the mathematical concepts involved in the task. A similar level of SCK was also necessary for Ms. Skywalker to scaffold tasks over time, allowing subsequent tasks to build upon the ideas introduced in previous ones.

One strategy that was *not* reliant upon Ms. Skywalker’s *MMRE* experience was the macro strategy of a *conducive classroom environment*. Since no particular mathematical content knowledge is necessary to deemphasize performance, establish helpful logistics and procedures, learn individual students’ strengths and weaknesses, and create a fun classroom atmosphere, this strategy could in theory be implemented by any interested teacher. However, it is unlikely that this macro strategy, apart from the other two, would suffice to establish the productive sociomathematical norms. Williams and Baxter (1996) observed a seventh-grade math class that was “friendly and warm” (p. 28) and where “most students trusted and respected the teacher” (p. 28). However, they also noted that the class’s discourse often “did not support… the creation of useful mathematical knowledge” (p. 36). Likewise, Nathan and Knuth (2003) documented a sixth-grade classroom where a central goal of the teacher was for “students to value all of the opinions expressed in class” (p. 186), but the corresponding discourse often had “a lack of rigorous argumentation and evidence” (p. 198). Together, these studies support the notion that a *conducive classroom environment*, while certainly important, is not by itself sufficient to establish the productive sociomathematical norms.
Therefore, I conclude that the norms and many of the teacher strategies identified in this study were dependent on Ms. Skywalker’s experience with MMRE. In other words, I believe that if Ms. Skywalker had not participated in MMRE professional development, she would not have been able to utilize the same strategies to successfully establish the same sociomathematical norms. However, it was not exactly Ms. Skywalker’s experience with MMRE that enabled her to achieve the results described in this study. More precisely, it was the increased mathematical confidence that she obtained from the professional development, the skill of considering multiple perspectives, and the increased SCK that allowed her to achieve the results described in this study. If a different teacher did not have this level of confidence, depth of content knowledge, and skill in considering multiple perspectives, it is doubtful whether they would be able to effectively employ the same strategies and experience the same success as Ms. Skywalker did. A possible direction for future research would be to investigate what strategies a teacher could employ who wanted to establish the norms from phase one but lacked an extensive professional development experience like MMRE.

Having discussed the dependence of the results upon Ms. Skywalker’s qualifications, I now consider their potential dependence upon the particular group of students in the case study classroom. In other words, could a teacher with Ms. Skywalker’s qualifications employ these strategies and expect similar results in their own classroom? Was anything about the case study class unusual or atypical? I asked Ms. Skywalker this question and she informed me that her 2014–2015 class had more “highly-capable” students than was typical in her previous years of teaching. She explained that in previous years, she typically had 2 highly-capable students in her class. During the 2014–2015 year, she had 8 such students out of a class of 26. However, only 2 of these students qualified as highly-capable in mathematics; the other 6 had qualified as highly-capable in language arts. Ms. Skywalker noted that while

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22 I believe that even without MMRE, Ms. Skywalker would have been able to successfully establish active listening since this social norm does not require mathematical content knowledge or mathematical confidence to understand or participate in.

23 As was explained in chapter 3, “highly-capable” students are those who, after testing, have been shown to be at or above the 97th percentile for academic performance relative to statewide averages.
these highly-capable students did sometimes make mathematical reasoning easier to elicit, they also introduced some additional challenges:

Ms. Skywalker: Now, if you want to be truthful, my high-cap kids have been some of the hardest to get to explain their thinking.

Many of the highly-capable students seemed to begin the year with strong performance-oriented views of mathematics and also seemed to be the most resistant to changing these views. Ms. Skywalker believed that these students had been rewarded in previous grades for obtaining answers quickly and had consequently formed their classroom identities as “the smart kids,” where “smart” meant getting answers faster than peers. Because of this, many of them did not initially value discourse about mathematical reasoning, with a few even expressing disdain for such discourse, as in the following instance:

Ms. Skywalker: Arnold was sitting there the first couple weeks [of the school year] going (makes an annoyed expression). And I said something like, “You know what we do.” [And he said], “Yeah, not until this year I didn’t know what we do, but apparently we explain our thinking.”

Peter: That’s what he actually said?

Ms. Skywalker: Oh, he actually said that to me! And I’m like, “Yeah. We do.”

Because of these additional challenges, Ms. Skywalker did not believe that eliciting student reasoning was necessarily easier overall with highly-capable students. She thought that the challenges were simply different. Furthermore, it was not merely the highly-capable students who demonstrated evidence of the norms. Nearly all of the students in the classroom evidenced the norms, indicating that Ms. Skywalker’s strategies to establish these norms were effective for a variety of students and not just her highly-capable ones. Therefore, while the composition of Ms. Skywalker’s 2014–2015 class was somewhat atypical due to the number of highly-capable students, I assert that her strategies were not dependent upon this particular dynamic and would have been effective even for a more “normally composed” classroom. With this said, however, one further point requires clarification. Part of Ms. Skywalker’s conducive classroom environment macro strategy was to learn individual students’ strengths and weaknesses. Although the effects of this sub-strategy were unapparent through observations, Ms. Skywalker informed me during
interviews that this knowledge of her students guided many of her interactions with them, influencing who she called on in certain situations, how she corrected mistakes, how she dealt with misbehavior, and how she sequenced student work. As she stated in her own words, “You work with what you have.” Given a different group of students, Ms. Skywalker would likely adjust her strategies based on her knowledge of them. For example, if a particular class had an unusually high percentage of students lacking mathematical confidence, then Ms. Skywalker might spend noticeably more time at the beginning of the year on her conducive classroom environment and foundational skills macro strategies than what I witnessed during the 2014–2015 school year. While it is certainly reasonable to expect that these year-to-year adjustments would not constitute a radical change in her pedagogy, the exact variation that might occur from class to class is impossible to determine apart from actually observing Ms. Skywalker teaching multiple classes. Thus, while asserting that Ms. Skywalker’s strategies are not dependent upon her 2014–2015 class, I also recognize that some amount of variation in her strategies will likely occur from class to class as an expression of her conducive classroom environment sub-strategy of learning individual students’ strengths and weaknesses.

The relationship between macro and micro strategies

In the previous chapters, Ms. Skywalker’s macro and micro strategies were presented separately, the micro strategies in chapter 4 followed by the macro strategies in chapter 5. But another question that naturally arises concerns the relationship between these two types of strategies. How did they support and complement each other? Because they were smaller in scope and more narrowly-focused, the micro strategies can be thought of as imbedded within the macro strategies. In other words, as Ms. Skywalker was working on establishing a conducive classroom environment, foundational skills, and employing a concept-oriented task philosophy, she was using the micro strategies repeatedly. In this way, the macro strategies helped to both guide and augment the micro strategies. These two points will be elaborated upon in the upcoming paragraphs.
First, the macro strategies helped to guide the use of the micro strategies. Since they provided a longer-term vision for the progression and development of the class, the macro strategies functioned as a sort of map, focusing Ms. Skywalker’s attention on strategic ideas and topics. This, in turn, narrowed down the infinite range of possible actions for Ms. Skywalker, allowing her to use the micro strategies in a purposeful and focused manner rather than a haphazard one. For example, during the second week of the school year, Ms. Skywalker made the following comment to her class de-emphasizing performance:

Ms. Skywalker: If you don’t understand it, your goal should be to learn it! We’re about growing and learning. We’re not about getting the right answer. Okay? This isn’t a competition. No one wins by being done first. Do you get that? There’s no bonus points for being done first!

From a micro-strategy perspective, this teacher action was a normative comment. From a macro-strategy perspective however, this normative comment was merely one step towards creating a conducive classroom environment. Ms. Skywalker knew that many of her students viewed mathematics competitively and that this mindset was counterproductive to the norms she wished to establish. Hence, she strategically employed this normative comment before starting an activity in order to proactively dismantle students’ performance mentalities. Similarly, when engaging in mathematical tasks, Ms. Skywalker’s concept-oriented task philosophy guided her on when to directly prompt students and draw attention to certain student ideas. For example, when the class was discussing how to solve \(50 - n = 30\), Marty explained that he used the fact that \(30 + 20 = 50\) to determine the value of \(n\). Ms. Skywalker then drew attention to Marty’s use of addition to solve a subtraction problem:

Ms. Skywalker: What strategy did he use? He used addition, but this is subtraction! Come on, Marty! This is a subtraction problem and he used addition! Can he do that?

Class: Yes.

Ms. Skywalker: Why? Why can he do that?

Her concept-oriented task philosophy focused Ms. Skywalker on Marty’s implicit reliance of inverse operations. This then guided her use of the micro strategy of highlighting a positive example. Thus, in this
manner, the macro strategies guided the focus of the micro strategies. The micro strategies describe what Ms. Skywalker did whereas the macro strategies help determine why the micro strategies were employed.

Besides guiding the micro strategies, the macro strategies also helped to augment their effectiveness. For example, establishing a safe classroom environment increases the likelihood that students will participate in discourse (Wood, 1999; Wood, Cobb, & Yackel, 1991). If contributing ideas carries a risk of embarrassment or humiliation, then students will naturally become reluctant to share. But when such risks are removed, students will more willingly contribute, thus making direct prompts more effective and providing more examples for Ms. Skywalker to potentially highlight. Similarly, providing students with foundational skills also increases the likelihood that they will respond successfully to direct prompts. For example, consider the task of justifying multiplicative parity properties. To accomplish this, Ms. Skywalker’s students relied on the visual representation shown below in Figure 6-1.

Figure 6-1. The visual representation the class used to justify multiplicative parity properties.

This representation helped illuminate the mathematical structure of the situation, allowing students to recognize “pairs” within the product. Identifying these pairs then allowed students to understand and articulate why the product would always be even or odd. But now imagine if students had been lacking the skill of creating visual representations. Without any sort of visual representation, it is questionable whether they would have been able to recognize the mathematical structure in this situation. If they indeed could not recognize this structure, no amount of direct prompting from Ms. Skywalker would have enabled them to successfully justify the multiplicative parity properties. Hence, in this situation, the
foundational skill of creating visual representations allowed students to successfully respond to Ms. Skywalker’s direct prompts. Finally, the macro strategy of a *concept-oriented task philosophy* created more opportunities for Ms. Skywalker to employ the micro strategies. Maintaining a focus on concepts allowed connections to be made, claims to be justified, different computational strategies to be discussed, and multiple perspectives to be offered. These sorts of situations gave Ms. Skywalker opportunities to either directly prompt her students or model the relevant sociomathematical norm herself. This task philosophy also afforded students more opportunities to participate in the norms without elicitation, thereby giving Ms. Skywalker more opportunities to highlight positive behavior. Therefore, the macro strategy of a *conducive classroom environment* increased the likelihood that students would *choose* to respond to direct prompts and create positive examples for Ms. Skywalker to highlight. The macro strategy of *foundational skills* increased the likelihood that students would *be capable* of responding to direct prompts and creating positive examples. And the macro strategy of a *concept-oriented task philosophy* provided *more frequent opportunities* for Ms. Skywalker to employ the micro strategies and for students to respond to them.

So far, I have considered the effect of the macro strategies upon the micro strategies, noting that the macro strategies guided and augmented the micro. But is there a relationship in the opposite direction? Did the micro strategies affect the macro? To address this question, a more fundamental question must be considered first: What exactly *are* the macro strategies? The micro strategies are categories of specific, directly observable actions that Ms. Skywalker used when interacting with her students. By contrast, the macro strategies are more abstract. They are longer-term plans that only become apparent when Ms. Skywalker’s class is observed and considered over time. Nevertheless, despite their more abstract nature, the macro strategies manifested themselves on a day-to-day level through specific micro strategies that Ms. Skywalker employed. The following remark, made by Ms. Skywalker to her students, exemplifies this phenomenon of a macro strategy expressing itself through a micro strategy:
Ms. Skywalker: Math in the real world is not the right answer on demand! It’s can you communicate the answer, and how! How! Why! No one in your dad’s work is going to come in and go, “Square root of 34! Go!” That’s not how it works!... Have you ever felt slow in math? Guess what? It’s not a race.

On a macro level, this remark evidences Ms. Skywalker building a conducive classroom environment. On a micro level however, this remark is an example of a normative comment. Therefore, the remark simultaneously evidences both a macro and a micro strategy, or to be more precise, a macro strategy expressing itself through a micro strategy. This example typifies how the micro strategies related to the macro. As Ms. Skywalker employed her longer-term macro strategies on a day-to-day basis, she did so through direct prompts, normative comments, highlighting positive examples, and modeling. Thus, to summarize the relationship between the strategies, the macro strategies guided and augmented the micro, while the micro served as the means through which the macro strategies were realized.

In discussing the relationship the between the strategies, it was stated that the micro strategies were more concrete and easily observed while the macro strategies were more abstract. This raises another potential question: Are the macro strategies actual entities that exist in Ms. Skywalker’s mind, or are they my own interpretive creation as a result of thematic analysis of Ms. Skywalker’s actions? Based on interview data with Ms. Skywalker, I argue that the macro strategies do in fact exist as entities in her mind. Consider the following interview quote in light of the foundational skills macro strategy:

Ms. Skywalker: This idea of systematic, explicit instruction... that’s something that really stood out to me [during my teacher preparation program]. This idea that it’s not just, “Hey, what do we got! Throw stuff at kids!” This idea of systematically, explicitly introducing ideas to them... These tasks are engaging, if they have access to them. So how do give them access?... You start with 100s chart at the beginning of the year. What patterns do you see? Why do you think those are happening?

By “systematic, explicit instruction,” Ms. Skywalker indicates that she is purposeful and deliberate in when she introduces ideas to her students. She seeks to give students access to mathematical activity in the classroom by beginning with concepts and activities that are within their grasp. The following quote
further reinforces the idea that Ms. Skywalker thinks of mathematical activity in terms of its requisite skills:

Ms. Skywalker: “Critique the reasoning of others.” I think I’m stepping there. Listening to others is the first step. Now, comparing your thinking to someone else’s, or your rationale to someone else’s, that might be this next step.

Peter: And they have to get their reasoning out there so they can start critiquing it.

Ms. Skywalker: Exactly! So we have first steps first. We can’t get there yet. There’s several things we need to do first. But I think we’re building towards that. And that’s the other thing that we were talking about: you can’t just come in do these [Common Core Math Practices]. I mean, you can understand them all you want, but there’s some foundation things that need to happen…

Although in this particular quote, Ms. Skywalker speaks only of the Common Core Mathematical Practices, when considered along with her “systematic, explicit instruction,” it is reasonable to conclude that foundation skills is a macro strategy that exists in her mind. Now consider the macro strategy of a conducive classroom environment. Ms. Skywalker speaks specifically to this strategy in a statement that immediately follows the preceding interview quote:

Ms. Skywalker: I think one of [the foundational things] is… an environment where learning is exciting, it’s fun, it’s safe, and you have rich activities and there’s certain systems and procedures in place to get started that way. I think you need to know your students… I think all of those are foundation skills before you can do [the Common Core Math Practices]. You cannot have productive mathematical discourse… you can’t do any of those things without some foundation skills, if you will, by your kids.

Note all of the environmental factors that Ms. Skywalker mentions in this quote. Thus, there is evidence to conclude that she is consciously striving to create a certain classroom environment. Finally, consider the macro strategy of a concept-oriented task philosophy. The following two interview statements indicate that this strategy was also an entity in Ms. Skywalker’s mind:

Ms. Skywalker: If there’s mathematical reasoning going on [in a task], if kids are engaged and they see the utility of math, and they can make connections, isn’t that more powerful than, “I did a rich task without [all these qualities]”?
Ms. Skywalker:  I don’t have to go out and find richer tasks! We can make almost any task rich.

The first quote indicates that Ms. Skywalker specifically focuses on the mathematical reasoning and coherency that a task elicits, rather than mere task completion. The second quote indicates that she views task implementation as the feature that makes a task mathematically rich, rather than some inherent quality of the task itself. Thus, these quotes support the idea the Ms. Skywalker is consciously aware of maintaining a concept-orientation when implementing tasks. At this point however, it must be stated that these conclusions can only be drawn because of interview data. By observational data alone, it would remain unclear whether the macro strategies were entities that existed in Ms. Skywalker’s mind or were merely interpretive constructs of the researcher.

The potential applicability of the results to other grade levels

This investigation of productive mathematical discourse was conducted by using a case study approach with a fifth-grade classroom as the case. Another question that naturally arises concerns the applicability of the results to other grade levels. Are the norms and strategies identified in this study equally applicable to all grade levels, or are they somehow inextricably linked to the upper-elementary level? In addressing this question, I will first consider the norms followed by the teacher strategies.

Based on their definitions presented in chapter 4, active listening, coherency, justification, and multiple perspectives could all conceivably be employed at any grade level. Their definitions contain nothing specifically linking them to upper-elementary level content. Students from many different grades could actively listen to their peers, recognize the interconnectedness of mathematics, justify claims, and utilize multiple perspectives in ways that are age-appropriate and content-appropriate. In contrast to these four norms, computational strategies is somewhat linked to the upper-elementary level. As defined in chapter 4, it specifically focuses on how students perform multi-digit computations. These sorts of computations, particularly involving multiplication and division, receive strong emphasis in the upper-elementary grades. In subsequent grades, knowledge of multi-digit computations, while regularly utilized,
is often assumed and receives much less explicit attention. Thus, *computational strategies*, as defined in chapter 4, is less relevant to classrooms at a higher grade level.

The limited relevance of *computational strategies* to higher grade levels raises a new question: Could this norm be generalized somehow to become equally relevant at all grade levels? Is there some “essence” to *computational strategies* that is not limited to multi-digit computations? I argue that the “essence” of *computational strategies* can be summarized by the concept of *procedural fluency* as detailed in the National Research Council [NRC] (2001) publication, *Adding It Up*. The NRC summarizes *procedural fluency* as “skill in carrying out procedures flexibly, accurately, efficiently, and appropriately” (p. 5), including knowledge of “reasonably efficient and accurate ways to add, subtract, multiply, and divide multidigit numbers, both mentally and with pencil and paper” (p. 121). It also includes “knowledge of ways to estimate the result of a procedure” (p. 121). This description aligns well with Ms. Skywalker’s goals in teaching her class various computational strategies. As the following normative comment indicates, procedural flexibility was a key skill she wanted her students to obtain through learning different computational strategies:

Ms. Skywalker: The other reason we share [computational] strategies is because your go-to strategy might not be the best strategy sometimes… You might have to go away from your go-to strategy.

She emphasized to her students that the various computational strategies were also intended to aid their ability to do mental math:

Ms. Skywalker: When you want to have mental [computational] strategies for math is when you’re in the grocery store or doing some things when you may not have paper in front of you. That’s why we practice multiple mental strategies.

Finally, Ms. Skywalker also stressed the importance of estimating the results of a computation:

Ms. Skywalker: This [student] and this [student] know that. They knew that [the answer] made sense. You know why? Because before we do anything [computationally], we should do an estimation
During interviews, Ms. Skywalker elaborated that she taught estimating as a means to help her students perform computations more accurately, believing that this would help them avoid common mistakes such as misplacing a decimal. Thus, Ms. Skywalker’s goals in promoting computational strategies align closely with the NRC’s (2001) definition of procedural fluency as “skill in carrying out procedures flexibly, accurately, efficiently, and appropriately” (p. 5). Procedural fluency includes computational procedures used on multi-digit numbers, but also encompasses more advanced procedures found at higher grade levels. Therefore, I conclude that the norm of computational strategies can be generalized to the norm of procedural fluency in order to be made equally applicable and relevant across all grade levels.

Having discussed the applicability of the norms to different grade levels, I now consider the applicability of the teacher strategies. All four of the micro strategies are equally applicable at different grades. Teachers of all levels can use direct prompts, normative comments, highlight positive examples, and model in age-appropriate ways in order to promote desired norms. Likewise, all three of the macro strategies can be applied at various grades. Regardless of the age of the students, teachers can work to make the classroom environment conducive for desired norms to emerge. As Ms. Skywalker pointed out during interviews, even adults require a safe environment:

Ms. Skywalker: [Teachers] are really fearful of basically showing that they’re stupid. They’re just like our students! They don’t want people to think that they’re dumb.

At all levels of mathematics, teachers can use tasks and examples in a way that emphasizes the underlying mathematical concepts. All students will require some sort of foundational or pre-requisite skills to succeed at their current grade level, so teachers can also work to systematically build these. Therefore, all three macro strategies are equally relevant at all grade levels.

**Implications**

In this section, the results of this study will be compared with the existing research literature. In the process, implications for researchers, teachers, and teacher educators will be discussed.
All five norms identified in phase one of this study have also been identified in numerous other studies as contributing positively towards mathematical discourse. Staples (2014), Hufferd-Ackles, Fuson, and Sherin (2004), McClain and Cobb (2001), Yackel, Rasmussen, and King (2000), Wood (1999), and many other authors point out the importance of justification, both for the discipline of mathematics itself and for mathematical discourse. Nathan and Knuth (2003), Sherin (2002), McClain and Cobb (2001), and Yackel, Rasmussen, and King (2000) note the importance of active listening, multiple perspectives, and computational strategies. Finally, the importance of establishing mathematical coherency has been highlighted by Corey, Peterson, Lewis, and Bukarau (2010), the National Research Council (2001), and Yackel, Rasmussen, and King (2000). Many of these same sources also recognize the micro strategies that Ms. Skywalker used: direct prompts, normative comments, highlighting positive examples, and modeling.

Macro strategies

While the norms and micro strategies that Ms. Skywalker employed are already well-recognized in the mathematics education literature, the idea of macro strategies offers new insight. The very concept of a “macro strategy” has not been well articulated in other longitudinal case studies. Studies that document a class’s development on a “macro level” often do not consider teacher strategies on the same macro level. For example, in their yearlong study of a third-grade classroom, Hufferd-Ackles, Fuson, and Sherin (2004) developed a framework to describe the discursive trends of the class over time, thus allowing them to consider discourse from a longitudinal perspective. However, when discussing the teacher’s role in this development, they focused on in-the-moment micro strategies, such as the nature and intent of direct prompts. Other longitudinal case studies allude to presence of macro strategies, but these strategies often do not receive much explicit attention. In her yearlong study of an eighth-grade teacher, Sherin (2002) examined her classroom on both a macro and micro level. On the “macro level,” she studied whether the teacher’s focus was on the process of how students talked about mathematics or the mathematical content of what students said. She noted that for the first several weeks of school year, the
teacher focused entirely on establishing norms for how the students should interact with each other. However, Sherin noted that these initial classroom activities were non-mathematical, thus indicating that the teacher was focused solely on establishing general social norms rather than sociomathematical norms. After several weeks, the teacher then introduced mathematical activity to the classroom. Although Sherin never provided a specific label for it, this progression of classroom activity indicates a macro strategy: establish social norms before engaging in mathematics. In her yearlong case study of a second-grade class, Wood (1999) investigated how the teacher helped her students to learn the practices of mathematical argumentation. During the initial days of the school year, the teacher explained her expectations of how students were to justify their assertions and indicate disagreement. In the following weeks, the teacher gradually extended her expectations, particularly regarding active listening, as the students practiced argumentation. Similar to Sherin’s (2002) study, this progression of teacher actions implies a macro strategy: gradually extend expectations for argumentation as students are able to practice them. However, Wood never labels this potential macro strategy with a concise term. Recognizing an explicit term such as “macro strategy” may help mathematics education research by drawing attention to what teachers are doing over time in their classrooms and if these actions reflect longer-term goals and intentions that would not be apparent from observing only a few episodes.

Ms. Skywalker’s macro strategy of decomposing mathematical practices into their constituent skills and then systematically establishing these skills is a strategy that has received relatively little attention within the mathematics education literature. McClain and Cobb (2001) addressed this idea somewhat in their investigation of sociomathematical norm formation in a first-grade classroom. They found that once students understood appropriate criteria to delineate between different solution strategies, they quickly began to comparing strategies, deeming some as efficient or easy. Thus, McClain and Cobb’s study indicates that the ability to distinguish different strategies is a foundation skill for comparing different strategies. Ghousseini and Herbst (2016) have stressed the importance of decomposing mathematical practices as a means to help pre-service teachers personally make sense of them. However, the case of Ms. Skywalker implies that teachers can use such decompositions not only to
personally make sense of mathematical practices, but also as a pedagogical tool to help plan the progression of mathematical activity in their classroom, especially near the beginning of the school year. Given the current prominence of the Common Core State Standards, this approach to mathematical practices is especially relevant. The eight mathematical practices included in the Common Core are complex and assume that students have numerous foundation skills. Decomposing these practices into more basic skills might help provide insight and guidance for teachers on the process of successfully implementing them.

**Task philosophies**

Ms. Skywalker’s concept-oriented task philosophy, especially when compared with other case studies, raises intriguing questions about the relationship between a teacher’s content knowledge and their task philosophy. For example, Cohen (1990) observed a teacher, Mrs. Oublier, who used many tasks with potential for illustrating and exploring various mathematical concepts. Unfortunately, this potential remained unrealized due to Mrs. Oublier’s “modest” (p. 322) content knowledge. Rather than highlighting key mathematical concepts, Mrs. Oublier instead focused on engaging all of her students in doing the various task activities. In other words, Mrs. Oublier’s task philosophy was task-oriented: engaging in and completing the task seemed to be her primary goal. Important mathematical ideas represented by the task were left implicit. Williams and Baxter (1996), Nathan and Knuth (2003), and Clement (1997) all described teachers whose main goals for their mathematical tasks seemed to be social in nature. Williams and Baxter (1996) noted that their teacher emphasized students working in groups and presenting their work to the rest of the class. Based on observations, they concluded that for at least some of the students, “discussing mathematics with other students rather than understanding mathematics was the purpose of group work” (p. 34). Nathan and Knuth’s (2003) teacher sought out tasks that encouraged student participation and student-led discussions. In a reflection however, the teacher realized that she often focused on student-to-student interaction and “wasn’t always thinking about the math” (p. 200). Clement’s (1997) teacher stated that her goals for her mathematical tasks were to engage students, cause
them to ask questions, listen to each other, and build upon each other’s ideas. While the goals from these various classrooms are all arguably commendable, they are not mathematical in nature. Student interaction, student presentations, group work, and asking questions are all social goals that do not specify how, if at all, students are interacting with mathematical ideas. Therefore, these teachers’ task philosophies were *socially-oriented*. In all three of these cases, the researchers indicated that the teacher’s mathematical content knowledge was less than ideal. Nathan and Knuth (2003) reported that their teacher’s content knowledge “was lacking in some major areas” (p. 181), Williams and Baxter’s (1996) teacher “expressed some concern regarding her ability to teach the mathematical concepts” (p. 28), and Clement’s (1997) teacher saw no inherent difference between mathematical discourse and discourse from any other subject area. By contrast, Ms. Skywalker’s task philosophy was *concept-oriented*. Rather than trying to complete tasks or foster certain social interactions, her overriding goal was to highlight the key mathematical ideas underlying the tasks’ activities. In order to do this however, Ms. Skywalker had to possess the necessary SCK to understand what the main mathematical ideas of the task were and how they related to other mathematical topics and ideas. Staples (2007) also observed a teacher who appeared to hold a concept-oriented task philosophy. She noted that “an overarching theme in [the teacher’s] work was that she fostered students’ thinking about the problem and not their progression towards task completion” (p. 33). Furthermore, Staples also noted that the depth of her teacher’s content knowledge was “quite remarkable” (p. 36). Collectively, these studies lend support to the idea that lacking SCK, teachers tend to adopt nonmathematical goals for their tasks. However, additional research is needed to substantiate this claim. Much research into mathematical tasks over the past two decades has been shaped by Stein, Grover, and Henningsen’s (1996) cognitive demand framework. This framework categorizes tasks based on the level and type of mathematical thinking required of students. While the framework is certainly helpful for analyzing the nature of a task itself as well as the teacher’s implementation of a task, it does not necessarily direct attention to the teacher’s intentions or goals for a task. For instance, based on a study of 213 middle school teachers, Wilhem (2014) found that teachers with less mathematical content knowledge were more likely to lower the cognitive demand of a task during implementation. However,
this does not address why teachers do so. It would certainly be reasonable to conjecture that teachers with less content knowledge, such as Mrs. Oublier, tend to adopt nonmathematical goals for their tasks and then, as a result, lower the cognitive demand of the task. For example, if a teacher has a task-oriented task philosophy, they would likely be willing to lower the cognitive demand if students’ progress was too slow so that the task could be completed in a timely manner. Likewise, if a teacher has a socially-oriented task philosophy, they might neglect or even intentionally lower the cognitive demand in their efforts to promote certain social behavior among students. However, such relationships are merely conjecture. More research is necessary to investigate the interaction between teacher content knowledge, task philosophy, and the cognitive demand of task implementation.

**High-level tasks**

Ms. Skywalker’s concept-oriented task philosophy also contains implications for both teachers and researchers on the nature of “high-level tasks.” The assumption is commonly made within mathematics education that high-level tasks, tasks that engage students in complex, non-routine mathematical thinking, must be long and somewhat ambiguous in their setup. For example, Stein, Grover, and Henningsen (1996), in their seminal work on the levels of cognitive demand, stated that:

> High-level tasks are often less structured, more complex, and longer than tasks to which students are typically exposed… Students often perceive such tasks as ambiguous and/or risky because it is not apparent what they should do and how they should do it. (p. 462)

Wilhelm (2014) implied that a single class period of 45 minutes may not provide enough time to complete a high-level task. Munter (2014) observed that most high-level tasks are often characterized by three phases: the launch phase, where the teacher explains the task, the explore phase, where students are given time to investigate the problem, and the summarize phase, where the class reconvenes to discuss the task in whole-class discourse. All of these authors imply that achieving high-level, non-routine thinking is likely to require more class time than traditional mathematical activity. In Ms. Skywalker’s class however, much high-level mathematical thinking was accomplished through relatively short activities.
For example, when a student mentioned in passing that 432 was even, Ms. Skywalker asked how it could be even when it contained an odd digit in the tens place. This then led to a discussion about the relationship between digits, place value, and parity. In the process of explaining why 432 is even, students justified why the ones place determines the parity of a number and clarified key distinctions between digits and place value. In another example, a student gave multiple answers of $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{4}{8}$ to the same problem. Ms. Skywalker then asked how these three fractions could be the same. This led to a discussion about the meaning of fraction notation and equivalency. Students ended up articulating key relationships and distinctions between digits, fraction notation, division, and the identity property of multiplication. In these two instances, high-level mathematical thinking was elicited from students, yet both episodes lasted roughly ten minutes. By Stein, Grover, and Henningsen’s (1996) description, these tasks were high-level, yet were relatively short and focused. Students did not perceive them as ambiguous because they understood what was being asked of them. The case of Ms. Skywalker implies that teachers and researchers alike could investigate how high-level mathematical thinking can be elicited from shorter, more focused tasks and activities.

**Tasks and mathematical practices**

Ms. Skywalker’s *concept-oriented task philosophy* contains further implications for researchers to investigate the relationship between students’ content knowledge, their experience with mathematical practices (i.e. *using* their content knowledge), and their ability to engage in high-level mathematical tasks. Teachers and researchers alike have given much attention to task selection and implementation in light of students’ prior mathematical knowledge. For example, Henningsen and Stein (1997) point out that a task that is too far removed from students’ prior mathematical knowledge will limit their ability to engage in high-level reasoning. To be successful, tasks need to build appropriately on students’ current level of mathematical content knowledge. However, comparatively little consideration has been given to selecting tasks in light of students’ proficiency with the requisite mathematical practices. For example, assume that a high-level task requires students to justify. Even if students have the necessary mathematical content
knowledge, they may have little experience with the actual practice of justifying. This lack of proficiency with a mathematical practice could potentially limit students’ ability to draw on and use their content knowledge. In his investigation into the process of proving, Karunakaran (2014) found that expert provers used similar content knowledge as novice provers but in more sophisticated ways. He suggested that the experts’ additional experience with proving allowed them to access and retrieve their content knowledge with greater ease than the novices. This supports the notion that researchers and teachers need to draw a distinction between students’ content knowledge and their experience using that content knowledge to engage in mathematical practices such as generalizing, justifying, interpreting notation, and creating visual representations. Through many of her tasks during the first month of school, Ms. Skywalker allowed her students to practice using their content knowledge to generalize, justify, and establish mathematical coherency. By choosing familiar and comfortable mathematical topics, she allowed her students to focus solely on these practices without being encumbered by a lack of content knowledge. Ms. Skywalker’s success in establishing mathematical practices in her classroom implies that researchers should investigate in greater detail the relationship between students’ content knowledge and their experience with mathematical practices (i.e. using their content knowledge) and how each of these areas allow them to engage in high-level mathematical tasks.

The constructivist dilemma revisited

The constructivist dilemma was introduced and discussed in detail in chapter 1. Interestingly, both observational and interview data did not indicate that the constructivist dilemma was present in Ms. Skywalker’s classroom. She was certainly successful in involving her students in meaningful mathematical discourse. As the norms from phase one indicate, the students regularly justified, established mathematical coherency, discussed computational strategies, and offered their own perspective on whatever topic the class was discussing. Throughout my frequent interviews with her, Ms. Skywalker never indicated that she felt any tension between student discursive participation on one hand and rigorous mathematics on the other. Why was this dilemma, so common in other case studies where
student discourse was a focus, unapparent in Ms. Skywalker’s classroom? I argue that the presence, or absence, of the constructivist dilemma is a direct result of how the teacher personally frames the issue of student discourse.

Many teachers and researchers have framed discourse by its *quantity* and its *form* (e.g. student-to-student, student-to-teacher, teacher-to-student). Examples of this can be found in Hufferd-Ackles, Fuson, and Sherin (2004), Nathan and Knuth (2003), Sherin (2002), Silver and Smith (1996), Williams and Baxter (1996), and Yackel, Cobb, and Wood (1991). This common method of framing discourse resulted from the influence of the 1989 and 1991 NCTM *Standards and Professional Standards*, which contained statements such as, “Teachers must do more listening, students more reasoning” (NCTM, 1991, p. 36). As Williams and Baxter (1996) have pointed out, the NCTM reform documents certainly “[gave] the impression that students’ talk is of comparatively greater value than teachers’ talk” (p. 23). However, framing discursive issues by the *quantity or form* of discourse says nothing about the *content* of that discourse. Many teachers have succeeded in increasing the quantity of *unproductive* student discourse (Nathan & Knuth, 2003; Chazan & Ball, 1999; Williams & Baxter, 1996). Furthermore, framing the issue in this way offers little guidance for teachers. A natural teacher response is to decrease their own participation in discourse and refrain from dispensing information in order to allow students a more active discursive role. Williams and Baxter (1996) go so far as to call this a “reasonable interpretation” (p. 23) of the NCTM documents. However, such negative guidance is unhelpful for teachers. It tells them what *not* to do, but fails to provide a positive vision of what to do, leading to teacher frustration and a loss of self-efficacy (Chazan & Ball, 1999; Smith, 1996). Because of the discursive void left by a passive teacher, the *quality* of classroom discourse often suffers. This in turn creates and reinforces a dichotomy in the teacher’s mind that places student discourse and rigorous mathematics at odds with each other. Hence, framing discourse in terms of *quantity and form* tends to lead to the constructivist dilemma.

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24 Chapter 1 contains a defense of the assertions that the NCTM reform documents were highly influential and that they called for increased quantity of student discourse.
There was no indication that Ms. Skywalker personally framed discourse in terms of quantity or form. Her role in classroom discourse was varied. At times she played a more passive role, insisting that her students supply ideas. However, she also frequently assumed an active role, giving no sign of hesitation or reluctance in the process of doing so. While she talked at great length during interviews about her students and her goals for them, she never made any comments about trying to increase student discourse in a general, unspecified sense. Rather, she spoke often about getting her students to participate in specific mathematical practices such as justification, making connections between different representations, and using various computational strategies. Likewise, I never observed her during class urging her students to talk more in a general, unspecified sense. Rather, she frequently encouraged them to engage in mathematical practices, such as the four practices that were identified in phase one as normative. For these reasons, I conclude that Ms. Skywalker framed discourse in terms of student engagement in mathematical practices, and that this subsequently allowed her to avoid the constructivist dilemma. Why would focusing on mathematical practices avoid this common discursive problem? A focus on student engagement in mathematical practices naturally led Ms. Skywalker to consider the content of student discourse rather than the sheer quantity of it. This, in turn, allowed her to avoid the commonly-employed teacher strategy of minimizing her own role in discourse in an attempt to maximize her students’ role. At times, she persistently pressed students for their reasoning, showing almost a tenacious insistence that certain ideas be student-voiced. However, she also utilized direct instruction, relying on it periodically to highlight conceptual connections, introduce ideas, and clarify confusion. In many cases, this direct instruction actually enabled student participation in mathematical practices. For example, Ms. Skywalker used direct instruction to introduce non-standard multiplication algorithms to her students in a fairly procedural manner. She then had students break into small groups to justify why these various algorithms worked. After a period of investigation, the groups then shared their insights with the rest of the class. Thus, Ms. Skywalker’s active role in direct instruction ultimately allowed students to engage in mathematical reasoning and practices. Rather than minimizing her own role, Ms. Skywalker continually adjusted the prominence of her role as necessary to assist student involvement in
the desired mathematical practices. Therefore, based on the case of Ms. Skywalker, I assert that framing student discourse in terms of *mathematical practices* is more helpful than framing it in terms of quantity or form of discourse. Successfully engaging students in mathematical practices, such as generalization and justification, naturally increases the quantity of student discourse while simultaneously ensuring that this discourse contains meaningful mathematical content. Additionally, framing discourse this way also helps to provide a more definite vision for the teacher’s role. Rather than simply being told what *not* to do, teachers can work towards instructing and supporting their students’ participation in these mathematical practices.

Findings and conclusions by other researchers support this alternative way of framing discourse. The second-grade teacher in Wood’s (1999) longitudinal study focused on teaching her students the practice of mathematical argumentation. Rather than increasing student discourse generally, this teacher taught her students to focus on mathematical ideas when listening and disagreeing and how to justify assertions. Far from observing the constructivist dilemma occurring, Wood (1999) commented based on subsequent observations that “the children appeared to have understood the teacher’s expectations” (Wood, 1999, p. 188). Nathan and Knuth (2003) observed the constructivist dilemma in a sixth-grade classroom where the teacher had focused on quantity and form of discourse. They concluded that “[good mathematical discourse] does not come about simply because the teacher creates the space for it” (p. 203) and suggested that talking *mathematically* should be viewed as a skill requiring explicit instructional attention. Staples (2007) pointed out that those interested in implementing the NCTM’s vision of reform often focus on “surface features” (p. 4) such as student-to-student interaction, rather than focusing on students’ interaction with mathematical ideas. Similarly, Staples (2014) also pointed out that many reformed-aligned pedagogies do not focus on fostering specific mathematical practices. These insights, supplied by other researchers, further support my assertion that discourse should be framed in terms of mathematical practices rather than quantity or form. A focus on mathematical practices draws attention to how students are interacting with mathematical ideas and helps illuminate the skills required to talk *mathematically*. However, the issue of how discourse is framed has received little explicit reflection or
discussion in the mathematics education literature. While researchers such as Nathan and Knuth (2003) and Staples (2014, 2007) have commented briefly on this topic, no articles have focused on summarizing the different ways teachers and researchers have framed discourse and the subsequent impacts on and implications for teaching. The case of Ms. Skywalker, combined with the insights of other researchers, implies that this topic requires greater awareness and explicit attention.

Finally, this study highlights the overall importance of context when investigating norms using fine-grained analysis of episodes. Compared to many other studies, this study is broad in its scope. For example, Cobb and Whitenack (1996) noted that they have performed extensive analyses on classroom episodes no more than fifteen or twenty minutes in length. Such fine-grained studies are certainly important in creating a more nuanced understanding of how norms emerge and affect classroom dynamics. This study, however, illustrates that contextual factors can substantially influence the effectiveness of teacher strategies. In the following interview, Ms. Skywalker compares the effectiveness of her direct prompts for multiple perspectives between the beginning and end of the school year:

Ms. Skywalker: Now I can walk around and say, “I want to see a number line, I want to see a table, I want to see a whatever,” and look what I get. Peter: You get all kinds of stuff. Ms. Skywalker: I didn’t get that at the beginning of the year, did I? [The students] have learned through seeing other people and seeing other things to try multiple representations. Remember before, I said, “I want to see 2 representations.” At the beginning of the year, they were like, “What do you mean? That’s the way I did it.” “Well, show me a different way.” “Umm, there’s only one way, it’s the way I did it. I did it the right way.” You know that! Those were seriously the conversations I was having with kids. Now I say, “I want to see at least 2 representations,” they all go (motion of holding up papers for teacher to see). Because they know! But it has not always been that way.

If a researcher had investigated the norm of multiple perspectives in Ms. Skywalker’s class by performing a fine-grained analysis of an episode near the beginning of the school year, they might arrive at a very different conclusion than if the same analysis had been performed on an episode from later in the school year. Hence, the chronological position of episodes within the school year may limit the conclusions that can be drawn from a fine-grained discursive analysis.
Methodological limitations

The methodology of this study was subject to several limitations. One such limitation stemmed from how I identified norms during phase one of the study. Recall that I identified norms by looking for unelicited student actions. These were any noteworthy actions, either discursive or non-discursive, that students performed without being explicitly prompted by the teacher to do so. While this allowed me to identify noteworthy things that students were doing, it did not allow me to identify noteworthy things that students were not doing. While some norms in the case study classroom were characterized by certain student actions, it is possible that other norms were characterized by an absence of certain student actions. Norms of this type would have been overlooked by my study. For example, Ms. Skywalker often mentioned during interviews how she worked to deemphasize a performance-oriented mentality about mathematics. As part of this, she did not allow her students to frantically wave their hands once they had an answer to one of her questions. During phase one of my study when norms were being identified, I did not typically see students frantically waving their hands. While potentially noteworthy, this was not recorded because it reflected an absence of student actions rather than a presence of them. To identify the noteworthy absence of student actions, I believe that a second “control classroom” would have been necessary for comparative purposes. This is because there were technically an infinite number of things that students in the case study classroom were not doing. They were not jumping up and down. They were not throwing pencils at each other. They were not screaming loudly. An endless list could be generated in this fashion, but how could one discern which items on it were noteworthy? Introducing a second, comparison classroom would allow for a more focused examination of noteworthy absences. One could then say that compared to the second classroom, the first classroom was lacking certain student actions.

Another methodological limitation came from how I defined and thought of norms themselves. I thought of norms as essentially specifying the outcome of a certain set of conditions, or an “if-then” relationship. In other words, if a certain set of conditions arises, then students will respond in the following manner. For example, the norm of justification means that if students provide an answer, then they will justify how they obtained it. The norm of computational strategies means that if students
mention a multi-digit computation, then they will share how exactly they performed it. The norm of multiple perspectives means that if a student initially shares their conceptualization of a certain situation or problem, then another student will volunteer their own conceptualization. Framing norms in this way clarifies what disconfirming evidence would look like. Disconfirming evidence would mean the presence of the specified conditions without the subsequent expected action. In other words, it would mean the “if” without the “then.” So for justification, this would mean that students share an answer without explaining how they obtained it. The norms of justification, computational strategies, and multiple perspectives were more “precise” norms. By “precise,” I mean that they each had a relatively clear set of conditions under which a certain student action could be expected. These conditions were given earlier in this paragraph. However, coherency and active listening were less precise norms because they did not have a clear set of corresponding conditions. Students in the case study classroom certainly demonstrated evidence of coherency: they would produce generalizations, recognize structural similarities, transfer knowledge from previous problems, and identify equivalencies. However, there was no clear set of conditions under which they would do these things. Similarly, students certainly showed evidence of active listening: they would use hand signals to indicate agreement with a statement and share their partner’s thinking after a pair-share. However, there was no clear set of conditions under which students would do these things either. They did not use hand signals after every statement uttered in the classroom. And they did not necessarily share their partner’s thinking after every pair-share. Hence, this means that for coherency and active listening, it was not possible to identify disconfirming evidence.

A final methodological limitation was the observation schedule: I did not observe Ms. Skywalker’s class every day and when I did observe, I only observed the class doing mathematics. Since I observed the class on 25 different occasions for a total of approximately 30 hours, it is reasonable to assert that I obtained an accurate sense of normal mathematical behavior. However, this assertion does assume that non-observed days followed the same expectations and patterns of activity as the observed days. From interviews, I discovered that many of Ms. Skywalker’s strategies were not confined to mathematics. Rather, she worked to promote the general ideas of justification and multiple perspectives
across her entire curriculum. In the following interview quote, she explains how she worked to promote justification within language arts:

Ms. Skywalker: And I really do think that the way I have approached math has helped science. It’s even helped reading. I don’t teach language arts the same way I used to. I make kids give me justifications for why they are making a certain decision. They didn’t do that before! It was all about authority. “Well this is the rule. This is the grammar rule. Grammar has rules.”

I witnessed some of how Ms. Skywalker elicited justifications within language arts. When students were correcting punctuation, they had to justify why a certain punctuation mark was appropriate (e.g. “This sentence needs a question mark at the end because it is a question,” or, “We need a comma here to separate the speaker tag from the quote”). Ms. Skywalker believed that promoting the same general norms across the entire curriculum, whenever feasible, would help establish them as expectations for mathematical activity. She explains this philosophy in the following interview quote:

Ms. Skywalker: MMRE has changed all of my practice. You see how [the students] use the same type of discourse and thinking in grammar with Greek and Latin roots.
Peter: Yeah, I remember that from the beginning of the year when you were talking about punctuation. “Why do we put that [punctuation mark] there?” It’s the same mindset all across the board, not just math.
Ms. Skywalker: Right. And I do credit MMRE with helping me think that way across the curriculum. So I reinforce [these expectations] in everything so that when we go to math, it’s not like a big switch.

She went on to explain how she tried to promote an expectation of justifying claims in all subjects. During science, she encouraged her students to connect what they were learning to previous knowledge and other areas of science. Whenever feasible, she encouraged students to offer their varying perspectives on the same topic. She also expected students to use hand signals to indicate active listening in all subjects. Hence, Ms. Skywalker promoted justification, coherency, multiple perspectives, and active listening in a general, non-mathematical way across all subjects that she taught.25 It would be

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25 Ms. Skywalker did not mention promoting computational strategies in other subjects. This is likely due to the inextricably mathematical nature of this norm. Justification, coherency, and multiple perspectives can all be
unreasonable to conclude that these efforts across the curriculum had no impact on mathematics.

However, due to practical constraints, I was unable to observe Ms. Skywalker’s class for the entire school day. Therefore, my report of her teacher strategies is necessarily limited. It is certainly reasonable to speculate that if Ms. Skywalker had only worked to establish her intended norms during mathematics, these efforts would likely have been less effective.

Although not necessarily a limitation, one other unintended feature of the methodology deserves mentioning. Ms. Skywalker purposefully did mathematics longer on the days that I observed her classroom. It was not uncommon for the class to spend anywhere from 1–2 hours doing math on these days. To compensate for this, Ms. Skywalker spent less time doing math on the days that I didn’t observe. She believed that overall, she spent approximately the same amount of time doing math as in previous years. However, this time was more concentrated within fewer days. These longer days aided her concept-oriented task philosophy, allowing her to often immediately pursue tangents that arose within a given task. If Ms. Skywalker had been constrained to doing mathematics for the same amount of time every day, she still would have been able to pursue tangents. However, she frequently would have had to wait until the next day to do so.

**Conclusion**

In light of the ongoing interest in student involvement in mathematical discourse within the field of mathematics education, as well as the continuing challenges in realizing it, this study sought to better characterize productive mathematical discourse and illuminate how a teacher might create it. To accomplish these goals, this study employed an in-depth case study of a fifth-grade classroom where the defining qualities of mathematically productive discourse regularly occurred. A collection of norms associated with mathematically productive discourse were identified, as well as teacher strategies used to establish these norms. These teacher strategies encompassed both smaller, day-to-day actions, as well as promoted in a more general, non-mathematical manner, but computational strategies unavoidably deals with mathematics.
larger, long-term plans spanning a period of weeks or months. While these norms and strategies were identified in a fifth-grade classroom, they may be applied or generalized across all different grade levels. Many of the strategies, however, require a certain depth of mathematical content knowledge on the teacher’s behalf and will remain inaccessible to teachers lacking this qualification. The results imply that teachers and researchers alike should consider teacher strategies on both a day-to-day scale as well as on a long-term scale. They also imply that high-level mathematical tasks may take the form of shorter activities, and that a teacher’s goals for their tasks may vary depending on content knowledge. Finally, the results imply that the way we choose to conceptually frame issues like mathematical discourse, whether done consciously or unconsciously, shapes the problems that we perceive and the solutions that we will attempt.
References


