The method of the local parabolic approximation for rough surface scattering

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(Received 8 June 1992; accepted for publication 7 June 1993)

A method for evaluation of scattering from rough surfaces which is similar to the Kirchhoff approximation is considered. However, it is based on a local parabolic approximation of the surface irregularities rather than a tangent plane approximation and two iterations of the surface field integral equation. The method, first proposed by Belobrov and Fuks (Izv. VUZ Radiofiz. 29, 1083-1089 (1986); Sov. Phys. Acoust. 31, 442-445 (1985)), accounts for local diffraction effects. Important modifications to the original method are introduced and extensive numerical results for Gaussian, one-dimensional randomly rough surfaces for both the Dirichlet and Neumann problems are provided. The validity of the local parabolic approximation is assessed by comparison with results from Monte Carlo simulations. It is demonstrated that the local parabolic approximation improves the Kirchhoff approximation for large and intermediate values of the surface correlation length especially in the backscattering region.

PACS numbers: 43.30.Hw, 43.30.Gv, 43.20.Fn, 43.30.Bp

INTRODUCTION

The basic idea behind the classical Kirchhoff method for rough surface scattering is to approximate the surface source density induced on a rough surface at point \( \mathbf{r} \) by the one induced on an infinite tangent plane located at the same point. Meecham showed that the Kirchhoff approximation is also the zeroth-order iterative solution to a surface field integral equation.\(^1\) This suggests that the Kirchhoff approximation can be improved by considering higher order iterations. However, such an effort can present serious problems. For example, recent work by Thorsson and Jackson showed that the second term of the iterative series solution—that is, the double scattering term—leads to a scattering cross section per unit length that increases as the surface length is increased.\(^2\)

Terms of the iterative solution to the surface field integral equation are represented by integrals over the scattering surface in which two regions contributing to the surface source density can be distinguished. The first region is the neighborhood of the point at which the source density is evaluated. Physically, the integral over this region accounts for local scattering—that is, directions and statistics of the local reflections and local diffraction effects, including local shadowing due to the existence of penumbra regions and local distortion of the scattered wavefront. The rest of the surface forms the second region which contributes to nonlocal scattering effects such as shadowing and multiple scattering. Various types of nonlocal effects are considered in Refs. 2-7. In this paper we are concerned with evaluating the contribution of the local diffraction effects only. For our purpose we employ an approach that consists of approximating an irregular surface locally by a parabola rather than by a tangent plane as in the Kirchhoff approximation. This method was developed by Belobrov and Fuks for the case of scattering from two-dimensional (2-D) randomly rough surfaces for both electromagnetic\(^3\) and sound\(^4\) scattering. It was shown that the parabolic approximation leads to an expression for the surface source density in terms of the expansion parameter \( \rho = (2k\delta\cos^2\theta)^{-1} \), where \( k = 2\pi/\lambda \) is the radiation wave number, \( \delta \) is the radius of curvature, and \( \theta \) is the local angle of incidence. An expression for the average field was derived for backscattering in the high-frequency limit from which the backscattering cross section was obtained. In this limit, however, diffraction effects are small (corrections to the scattered intensity start from second order in \( \rho \).

In the next section we give a detailed review of the parabolic approximation. In Section II, we develop closed form expressions for the coherent reflection coefficient and the incoherent bistatic scattering cross section within the framework of the parabolic approximation without resorting to the high-frequency limit. To facilitate comparison with Monte Carlo simulation results we restrict our problem to scalar-wave scattering from one-dimensional (1-D) surfaces (2-D scattering problem). In Section III we provide numerical results for a variety of surface statistical parameters and draw conclusions for the range of validity of the local parabolic approximation (LPA).

I. THE LOCAL PARABOLIC APPROXIMATION

Consider a scalar plane wave \( p_s(\mathbf{r}) = \exp(ik_1\cdot\mathbf{r}) \), \( \mathbf{r} = (x, z) \) incident on an infinite, one-dimensional randomly rough surface \( S \), Fig. 1. The two-dimensional incident wave vector is given by \( k_1 = k(\cos\theta_1, -\sin \theta_1) \) where \( \theta_1 \) is the incident angle and a harmonic time dependence of \( \exp(-i\omega t) \) is assumed. The surface \( S \) is defined by the

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random variable \( f=f(x) \) which is assumed to be differentiable. While the parabolic approximation is not restricted to particular surface statistics, the assumption of Gaussian statistics considerably facilitates the development of a closed form expression for the scattering cross section. Finally, we shall consider both Dirichlet (sound-soft) and Neumann (sound-hard) boundary conditions.

The starting point for our analysis is the surface field integral equations

\[
\frac{\partial \rho_D(r)}{\partial n} = 2 \int_S \frac{\partial G_0(r,r')}{\partial n} \frac{\partial \rho_D(r')}{\partial n'} \, dS', \quad (r,r' \in S)
\]

(1a)

and

\[
\rho_N(r) = \rho_D(r) + 2 \int_S \frac{\partial G_0(r,r')}{\partial n'} \rho_N(r') \, dS', \quad (r,r' \in S)
\]

(1b)

for the Dirichlet boundary condition and for the Neumann boundary condition, respectively, where the subscripts D and N stand for Dirichlet and Neumann, and \( \rho_D \) and \( \rho_N \) are the total fields at the surface. Equations (1a) and (1b) are obtained by taking the limit \( r \to S \) in the Green’s formula (see Ref. 1 or 10). It should be noted that the integrals on the right-hand sides are principal value integrals. The two-dimensional free-space Green’s function \( G_0 \) is proportional to the first-kind, zero-order Hankel function. For our purpose, however, the following integral representation of \( G_0(|r-r'|) \) is more convenient:

\[
G_0(|r-r'|) = -i \int_0^\infty \frac{dt}{4\pi t} \exp \left[ i(k^2 + i\epsilon)t + \frac{i}{4t}|r-r'|^2 \right].
\]

(2)

The parameter \( \epsilon > 0 \) insures satisfaction of the radiation condition and is eventually set to zero.

The solutions (the surface source densities) to (1a) and (1b) are sought in the form:

\[
\frac{\partial \rho_D(r)}{\partial n} = i(k_i \cdot \mathbf{n}) \exp(ik_i \cdot r) \psi_D(r)
\]

(3a)

and

\[
\rho_N(r) = \exp(ik_i \cdot r) \psi_N(r),
\]

(3b)

respectively. In (3), \( \mathbf{n} = \mathbf{n}(r) \) is the unit normal to the surface at point \( r \), and \( \psi_D \) and \( \psi_N \) are two unknown functions for which approximate solutions will be obtained below. Inserting (2) and (3) in (1) and performing a change of variables \( \rho = r' - r \), one obtains

\[
\psi(r) = 2 + \int_0^\infty \frac{dt}{4\pi t^2} \exp[i(k_i^2 + i\epsilon)t]
\]

\[
\times \int_S dS' \exp[i(k_i \cdot \mathbf{r} + \frac{i}{4t}r'^2)] K(\rho) \psi(r + \rho),
\]

(4)

where \( \psi(r) = \psi_D(r) \) and \( K(\rho) = K_D(\rho) = (i \lambda \cdot \mathbf{n})/(i \lambda \cdot \mathbf{n}) \) for the Dirichlet surface, and \( \psi(r) = \psi_N(r) \) and \( K(\rho) = K_N(\rho) = (i \lambda \cdot \mathbf{n}) \) for the Neumann surface. We note that depending on the particular approximate solution for \( \psi \), the inner integral in (4) may need regularization; see the text after Eqs. (8) below.

The zeroth iterative solution to (4) is the familiar Kirchhoff approximation

\[
\psi^{(0)}(r) = 2
\]

(5)

[see Eqs. (3a) and (3b)]. In the high-frequency limit the integral term in (4) vanishes and sound-hard and sound-soft surfaces are indistinguishable. Therefore, corrections to the surface source density introduced by higher-order iterations of (4) should in principle account for local diffraction phenomena, including local shadowing, non-local shadowing, and multiple scattering effects.

The first iteration is obtained by substituting (5) in the integral on the right-hand side of (4). When evaluating the first and second iterations, it is appropriate to introduce a local coordinate system \((\xi, \zeta)\) associated with the tangential and normal directions to the surface \( S \) at point \( r \). In this coordinate system the surface profile function in the small neighborhood of a regular point \( r \) is approximated by the parabola

\[
\phi(\xi) = \frac{1}{2} f'' \xi^2
\]

(6)

(hence, the name the local parabolic approximation) where \( f'' \) is the second derivative of the surface profile function with respect to \( \xi \). We remark that the same notation \( f \) is used to denote the surface profile function in both the global and local coordinate systems; the two systems are distinguished by the arguments \( x \) and \( \xi \), respectively.

In the local coordinate system \( |f''| = 1/\Re(\xi) \) (since \( f' = 0 \)) where \( \Re(\xi) \) is the local radius of curvature. It is convenient to measure all lengths in units of the wavelength \( \lambda = k^{-1} \). We also introduce the notation \( r' = (r_\xi, r_\zeta) = \rho/\lambda = k(\xi, f(\xi)) \) and \( x = r/\lambda \). The first iteration to \( \psi \) is then written

\[
\psi^{(1)}(x) = \int_0^\infty \frac{dt}{k^2 + i\epsilon} \exp[i(k^2 + i\epsilon)t] \int_{-\infty}^\infty \frac{d\tau_\xi}{4\pi} K(\tau_\xi)
\]

\[
\times \exp[i\tau_\xi \sin \theta \frac{k_\xi^2}{4t} + \frac{i}{2} \cos \theta \tau_\zeta^2(\lambda f''')],
\]

(7)

where \( \theta \) is the local angle of incidence measured from the normal at point \( r \), and the expressions for \( K \) are given by

\[
K_D(\tau_\xi) = \tau_\xi^2(\lambda f''') [1 + \tau_\xi^2(\lambda f''') \tan \theta],
\]

(8a)

\[
K_N(\tau_\xi) = -\tau_\xi^2(\lambda f''').
\]

(8b)
Several comments are in order. First, the integration with respect to \( \tau_2 \) is extended to infinity, which is justified when the major contribution to the scattered field comes from the vicinity of point \( r \). Additional corrections are needed to account for multiple scattering and nonlocal shadowing when they are important.

Second, the inner integral needs regularization at infinity which can be accomplished by initially assigning a small imaginary value to the wave number \( \lambda \). This is equivalent to assuming a medium with small loss; which is eventually set to zero. However, for simplicity we shall manipulate (7) in a formal manner which leads to the same final expressions for \( \psi^{(1)} \) and \( \psi^{(2)} \) as the regularized Eq. (7).

Third, anticipating that the considered approximation will be accurate for surfaces with an average radius of curvature much greater than \( \lambda \)—that is, \( \lambda/\mathcal{R} \ll 1 \), we note from (8a) and (8b) that to the leading order in \( (\lambda f''/\mathcal{R}) \), \( \psi^{(1)} \) and \( \psi^{(2)} \) differ only by a sign. As we shall see shortly, the leading order of the second iterate is \( (\lambda f'')^2 \); therefore, we relocate the contribution of \( \psi^{(1)}(x) \) that comes from the second term in (8a) to \( \psi^{(2)}(x) \).

Fourth, after performing the indicated integration the third term in the second exponent of (7) leads to a very complicated nonlinear functional dependence of \( \psi^{(1)} \) on \( f'' \), preventing further analytical development. In Ref. 9, this difficulty has been dealt with by expanding the exponent and retaining the zeroth- and first-order terms in \( (\lambda f'/\mathcal{R}) \). In this manner an additional contribution to the second iterative solution is included. We tried this also, but numerical results for the scattering cross section, which will not be presented here, were not as good as those obtained by simply neglecting the term in question. A plausible explanation for this behavior is that when the first order in \( (\lambda f'/\mathcal{R}) \) phase term is kept, an oscillating, bounded function is approximated by an unbounded quadratic in the \( \tau_2 \) function.

Our last comment is related to the form of (6). Considering only points for which (6) is true, we, in fact, neglect contributions from all inflection and higher-order critical points. The probability for such points to occur is relatively small,13 yet evaluation of their contribution to the scattering cross section is an interesting problem. However, it will not be pursued here.

To evaluate the integral with respect to \( \tau_2 \) in (7), we use a change of variables \( \tau_2 = 2 \sqrt{\mathcal{R}/\lambda - 2 \sin \theta / \lambda^2} \) and transform the \( \mu \)-contour along the positive real axis to a ray in the complex plane which starts from the origin at an angle of \( \pi/4 \). Integrating the result of the \( \mu \)-integration with respect to \( t \), letting \( \epsilon \to 0 \), and reverting to the global coordinate system, one arrives at

\[
\psi^{(1)}(r) = -\psi^{(1)}(r) = -\frac{i \lambda}{\cos \theta} \mathcal{R}(x),
\]

where \( \mathcal{R}^{-1}(x) = f''(x)/[1 + (f'(x))^2]^{3/2} \).

Note, that the first diffraction corrections to the surface source densities have a phase shift of \( \pm \pi/2 \) from the Kirchhoff approximation.

The second iterate is obtained by substituting \( \psi^{(1)} \) evaluated at the point \( r + \rho \) in the right-hand side of (4) (for the case of the Dirichlet surface, retaining the leading order of \( K_D \) only). As mentioned earlier, for a sound-soft surface the contribution from the first iterate combined with the second term of (8a) is used to find \( \psi^{(2)}(r) \). A technique similar to that used to obtain (9) yields

\[
\psi^{(2)}_D(r) = \frac{\lambda^2 (5 - 6 \cos^2 \theta)}{2 \cos \theta \mathcal{R}^2(x)}, \tag{10a}
\]

\[
\psi^{(2)}_N(r) = \frac{\lambda^2}{2 \cos \theta \mathcal{R}^2(x)}. \tag{10b}
\]

From (9) and (10) it is seen that up to second order the local parabolic approximation leads to expressions for the surface source density in terms of the expansion parameter \( \theta = (2k \alpha^2 \cos^2 \theta)^{-1} \) which has been called the Lynch parameter.14,15 The appearance of the \( \cos^2 \theta \) term in the denominator degrades the LPA when the mean values of \( \theta \) are close to \( \pi/2 \). This occurs when both the incident direction is away from the normal and the scattering direction approaches forward grazing.

II. SCATTERING CROSS SECTION

The transition matrix corresponding to the surface source densities obtained in the previous section (after integrating once by parts and keeping terms through order \( \psi^2 \)) can be written in the following form for both Dirichlet and Neumann surfaces infinite in extent:

\[
T(\theta_s, \theta_t) = \frac{k_s \cdot v}{2nk_e \rho_2} \int_{-\infty}^{\infty} dx \exp \{iv_x x + iv_y f(x)\} \times \left[ 1 + i2\mathcal{R}(x) + b\mathcal{R}^2(x) \right], \tag{11}
\]

where \( k_s = (k_{sx}, k_{sy}) \) and \( \theta_s \) are the scattered wave vector and the scattering angle measured from the horizontal, respectively, and \( v = k_t - k_s \). The expressions for the coefficients \( a \) and \( b \) are

\[
a_D = -a_N, \quad a_D = \lambda (2 \cos \theta)^{-1}, \quad b_D = a_D^2 (5 - 6 \cos^2 \theta), \quad b_N = -a_N^2. \tag{12a}
\]

Equation (11) is written by employing the traditional expression for the transition matrix for the case of 1-D surfaces (2-D scattering problem) (see, for example, Ref. 16).

Next, we replace the factor \( [1 + (f'(x))^2]^{-3/2} \) with an estimate \( [1 + \gamma^2]^{-3/2} \), where \( \gamma \) is the rms slope. For the local angle of incidence \( \theta \) we substitute its value at the specular point. This substitution is justified (for \( \theta \) in the amplitude) for both the Kirchhoff approximation and the so-called modified Lynch variational method.15 The legitimacy of such a substitution is an additional assumption in (11). With it, and incorporating the factor \( [1 + \gamma^2]^{-3/2} \), the coefficients \( a \) and \( b \) take the form:

\[
a_D = -a_N = -\frac{k_e^2 |v|^3}{2(k_s \cdot v)} \left[ 1 + (1 + \gamma^2)^{-3/2} \right], \tag{12b}
\]

\[
b_D = a_D^2 \left[ 5 - 6 \left( \frac{k_s \cdot v}{k_e} \right)^2 \right], \tag{12c}
\]

The form of Eqs. (12) readily ensures the symmetry of \( T(\theta_s, \theta_t) \) upon exchanging \( \theta_t \) and \( \theta_s \). This is an artifact of 2013. K. Ivanova and S. L. Broschat: Local parabolic approximation
the approximation for the local angle of incidence. Thus it seems unnecessary to use recently derived, manifestly reciprocal forms of the transition matrix such as Dashen’s formula\textsuperscript{18,19} in conjunction with the parabolic approximation.

The ensemble average of (11) is obtained by appropriately differentiating $\langle \exp[i\omega_{s}f(x)+i\beta f''(x)] \rangle$ with respect to the parameter $\beta$ and then setting $\beta=0$. This gives,

$$
\langle T'(\theta_s,\theta_t) \rangle = -\delta(k_{\alpha}-k_{\alpha})\exp[-\frac{1}{2}v_{\alpha}^{2}B_{r}(0)]
$$

$$
\times \{1-a_{0}[B_{r}''(0)+b[B_{r}''(0)]
$$

$$
-\nu_{\alpha}^{2}[B_{r}''(0)]^{2} \}.
$$

In (13) the correlation function $B_{r}(x)$ and its second and fourth derivatives evaluated at $x=0$ are present. The exponential factor is the Kirchhoff approximation coherent reflection coefficient; the expression in the braces represents corrections for diffraction effects included in the parabolic approximation.

$$
\sigma(\theta_s,\theta_t)=\frac{(k_{\alpha} \cdot \nu)^{2}}{2\pi kv_{\alpha}^{2}}\exp\{-\nu_{\alpha}^{2}B_{r}(0)\} \int_{-\infty}^{\infty} dx \exp[i\omega_{s}x]
$$

$$
\times \left[ \exp[v_{\alpha}^{2}B_{r}(x)]-1-2av_{\alpha}[B_{r}''(0)-B_{r}''(x)]\exp[v_{\alpha}^{2}B_{r}(x)]-B_{r}''(0) \right]
$$

$$
+a^{2}\left[ B_{r}''(x)+v_{\alpha}^{2}[B_{r}''(0)-B_{r}''(x)]^{2}\exp[v_{\alpha}^{2}B_{r}(x)]-\nu_{\alpha}^{2}[B_{r}''(0)]^{2} \right]
$$

$$
+2b\left[ B_{r}''(x)-v_{\alpha}^{2}[B_{r}''(0)-B_{r}''(x)]^{2}\exp[v_{\alpha}^{2}B_{r}(x)]-B_{r}''(0)+\nu_{\alpha}^{2}[B_{r}''(0)]^{2} \right].
$$

When $a$ and $b$ are set to zero, (15) readily reduces to the expression for the incoherent scattering cross section for the Kirchhoff approximation.

The numerical results presented in the next section are obtained for the case of a Gaussian correlation function of the form

$$
B_{r}(x)=h^{2}\exp[-x^{2}/l^{2}],
$$

(16)

where $h$ and $l$ denote the rms height and correlation length of the surface, respectively. For a Gaussian correlation function it is computationally more efficient to use a series representation of (15). The latter is given in the Appendix.

III. NUMERICAL RESULTS AND DISCUSSION

In this section we examine the accuracy of the local parabolic approximation, both first- and second-order, by comparing results for the scattering cross section with Monte Carlo simulation results.\textsuperscript{2,21,22} We begin by remarking that for a Gaussian correlation function (16), an estimate for the small parameter $\varphi$ is given by

$$
\varphi \sim 2\sqrt{\frac{kh}{(kl)^{3}}}[(1+\gamma^{2})(1+\cos(\theta_{s}-\theta_{t}))]^{-3/2}. \quad (17)
$$

In addition to $\langle T(\theta_{s},\theta_{t}) \rangle$ and its complex conjugate the expression for the incoherent scattering cross section involves the second moment of the transition matrix (see for example Refs. 20, 21)

$$
\sigma(\theta_s,\theta_t)=\frac{2\pi k_{\alpha}^{2}}{kL}\langle \langle T(\theta_s,\theta_t)T^{*}(\theta_s,\theta_t) \rangle \rangle
$$

$$
-\langle \langle T'(\theta_s,\theta_t) \rangle \rangle^{2},
$$

(14)

where $L$ is the length of the illuminated portion of the surface. The second moment requires the average of $\exp[i\omega_{s}[f(x_{1})-f(x_{2})]+i\beta_{1}f''(x_{1})+i\beta_{2}f''(x_{2})]$, where $x_{1}$ and $x_{2}$ are two distinct points. We differentiate this quantity with respect to the parameters $\beta_{1}$ and $\beta_{2}$ as many times as necessary to retrieve the various powers of $f''$ that appear in (14). At the conclusion $\beta_{1}$ and $\beta_{2}$ are set to zero. Without presenting the details, we write down the result for the incoherent scattering cross section $\sigma(\theta_s,\theta_t)$:

$$
SS(\text{dB})=10\log[\sigma(\theta_s,\theta_t)].
$$

(18)

The local parabolic approximation (LPA) should be accurate for small $\varphi$ and, thus, for relatively large $kl$ and small $kh$. However, as we shall see, moderate and small $kl$ present problems even when $kh<1$. In general, the LPA should perform better in the backscattering region compared to the forward one since, for fixed $kl$ and $kh$, $\varphi$ exhibits a minimum in the backscattering direction.

Figures 2-6 present numerical results for the scattering strength $SS$ as a function of the scattered angle. The relationship between the scattering strength and the scattering cross section $\sigma(\theta_s,\theta_t)$ is given by

$$
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$$
(PT) curve, whichever of the two classical methods is more appropriate for the particular surface parameters. The fluctuating curves denoted by IE (D) in Figs. 2–6 are the integral equation results for a Dirichlet boundary condition. Each well-defined peak in the specular direction is due to the coherent part of the scattered intensity which is excluded in Eq. (14).

Our first numerical example illustrates a case when \( k_l = 15 \) is large and \( k_h = 0.1 \) is small. The angle of incidence is \( \theta_i = 45^\circ \). From the exact integral equation result, it is seen that the Kirchhoff approximation is accurate for all scattering angles except low forward ones. Figure 2 shows that the first-order diffraction term of the LPA (LPA-1) overcorrects the KA; however, when the second-order correction is included, agreement with exact numerical results is excellent. Note that for the statistical parameters used in Fig. 2, \( \varphi \) as estimated by (17) has a maximum value of 0.014. Diffraction corrections to the

FIG. 2. Bistatic scattering strength versus the scattered angle (measured from the horizontal) for normalized rms height \( k_h = 0.1 \), normalized correlation length \( k_l = 15 \), and angle of incidence \( \theta_i = 45^\circ \). The curves correspond to the Kirchhoff approximation (KA), the first- and second-order local parabolic approximations for Dirichlet boundary conditions [LPA-1 (D)] and [LPA-2 (D)], respectively, the second-order local parabolic approximation for the Neumann boundary condition [LPA-2 (N)], and the integral equation for Dirichlet boundary conditions [IE (D)].

FIG. 3. The same as Fig. 1 except \( \theta_i = 20^\circ \) and the statistical parameters are altered as indicated.

FIG. 4. The same as Fig. 1, but \( k_l = 5 \) and the dotted curve represents the first-order perturbation (PT) result instead of the Kirchhoff approximation result.

FIG. 5. The same as Fig. 1 except the statistical parameters have been changed as indicated.

FIG. 6. Scattering strength as evaluated by KA, PT, IE, and the LPA involving the first iterate only [see Eq. (7)] for Dirichlet surfaces with statistical parameters as shown in the figure and with \( \theta_i = 45^\circ \).
KA for the case of the Neumann boundary condition increase the predicted scattered power in the forward region.

Extensive numerical tests of the LPA were carried out for cases for which the Kirchhoff approximation is accurate \((kl>6)\) away from low grazing angles.\(^2\) It was found that in the backscattering region the diffraction corrections associated with the LPA always improve the Kirchhoff approximation. The magnitude of these corrections decreases with increasing \(kh\), ranging from a few decibels (for \(kh<0.3\)) to zero as the high-frequency limit is approached \((kh>10)\). In the forward grazing region, however, the LPA performs well only for small (relative to \(kl\)) values of \(kh\). As \(\theta\) increases, the angular factor in \((17)\) increases; if the ratio \((kh)/(kl)^2\) is not small enough to compensate for this increase, \(\rho\) becomes large and the LPA is no longer accurate. A typical example is shown in Fig. 3 where \(kh=1.76, kl=7.46\), and the angle of incidence is \(\theta=20^\circ\). The integral equation curve extends only from 50 to 180 deg; for scattering angles below 50 deg the theoretical LPA result cannot be accurately compared with the IE result.\(^24\) Good agreement with Monte Carlo simulations is achieved for \(\theta<100^\circ\). In the forward region \(\rho\) becomes greater than unity for \(\theta>158^\circ\), and the LPA breaks down. This example provides a possible clue for understanding the controversial question of whether a shadowing function based on high-frequency concepts\(^22\) should be used for moderate \(kh\). The results for the backscattering region suggest that most of the shadowing is due to the appearance of local penumbra areas, and a shadowing function, if used, would lead to an overestimation of the shadowing effects.

The case illustrated in Fig. 4 involves the same parameters as those used for Fig. 2 except that \(kl\) (=5) is three times smaller; thus, the maximum value of \(\rho\) is increased nine times. The KA is not valid for this example.\(^22\) Comparison with the integral equation result shows that the second-order LPA result agrees very well for all scattering angles except for very low ones, \(\theta<23^\circ\) and \(\theta>175^\circ\). We note that, despite the small value of \(kh\) (=0.1), it was necessary to use fourth-order perturbation theory\(^21\) in order to achieve accuracy at these low angles. For the case of sound-hard surfaces, the diffraction terms result in higher levels of the scattering strength except near the specular direction.

In Fig. 5 the scattering strength for a surface with yet a shorter correlation length is depicted; \(kh=0.33, kl=2.83,\) and \(\theta=45^\circ\). Consider first the backscatter and specular parts of the curve. Again the first-order term overcorrects; for \(\theta<60^\circ\) the second-order term improves the result to some extent. To understand why the LPA is not as good for moderate \(kl\) as it is for large \(kl\), note that the magnitude of \((\lambda f'')\) for a Gaussian correlation function is \(2\sqrt{3}(kh)/(kl)^2\), whereas the magnitude of the dimensionless third derivative is \(|\lambda^2 f'''| \sim 8 \sqrt{2}(kh)/(kl)^3\). Thus, depending on an additional angular factor, the contribution of the neglected third derivative of the surface profile may become appreciable for moderate and, in particular, for small \(kl\). In the forward direction the agreement is, as expected, worst, although the first correction appears to behave well. That this is accidental is revealed when the second correction is included (the solid curve). The breakdown of the LPA at forward grazing angles is expected because of the angular factor in \((17)\). At low scattering angles, \(\rho\) rapidly increases and for \(\theta>177^\circ\) it exceeds unity.

The conclusion that the third- and higher-order derivatives are needed for moderate and small \(kl\) is supported by the results shown in our last figure. For this case, \(kh = 0.03, kl=1.5,\) and \(\theta=45^\circ\), and the first iterate of \((4)\) solved numerically, as well as the first-order perturbation result, are extremely accurate. Hence, stopping after the first two iterations only is not a possible source of error in the LPA. The dashed curve represents the first order in \((\lambda f'')\) correction in the first iterate [LPA-1 (D)-first order], whereas the solid curve is the small- and second-order corrections in the first iterate [LPA-1 (D)-second order], both for the Dirichlet boundary condition [see Eq. (8a)]. As expected, the contribution from the second iterate (not shown) and therefore from the higher-order iterates is negligible. Thus the only possible source of error comes from neglecting the third- and higher-order derivatives of \(f(x)\) in \((6)\).

IV. SUMMARY

To summarize, we have presented a method for evaluating scattering from randomly rough surfaces which we call the local parabolic approximation (LPA). This approximation, originally proposed by Belobrov and Fuks, is similar to the Kirchhoff approximation, but it includes local diffraction effects. We introduced two modifications to the original scheme: (1) developing an expression for the scattering cross section without resorting to a high-frequency approximation and (2) neglecting a first-order correction originating from the phase of the integrand of Eq. \((7)\). Our averaging procedure also involves replacing the factor \([1+(f''(x))]^{-3/2}\) in Eq. \((11)\) with an estimate \([1+y^2]^{-3/2}\) and using values of the local angles of incidence at the specular points. Both modifications \((1)\) and \((2)\) are very important in improving the LPA and are readily justified when the LPA is compared with Monte Carlo simulations. The inclusion of the local diffraction corrections in the theory makes the LPA more accurate than the Kirchhoff approximation. The LPA improves the Kirchhoff approximation when the latter is considered to be accurate, but in addition it is accurate for cases outside the range of validity of the KA, namely when \(kl<6\).

The LPA method is more accurate in the backscattering directions than in the forward ones. It breaks down in the forward scattering region if the ratio \((kh)/(kl)^2\) is not small enough to ensure \(\rho<1\) [see Eq. \((17)\)]. It is shown that for \(kl<3\), contributions from terms involving derivatives of the surface profile function higher than the second are important.
ACKNOWLEDGMENTS

This work was funded by the National Science Foundation ECS 9058186 and the Office of Naval Research Code 11250A. The authors wish to thank Dr. Eric Thorsos for providing the Monte Carlo results for Figs. 2-6, and Prof. Fuks and Dr. Oleg Yordanov for comments and discussions. One of us (K.I.) would like to express her gratitude to the Bulgarian Ministry of Science and Education for support under contract F4.

APPENDIX

In this Appendix we provide an alternate form of the scattering cross section with a Gaussian correlation function (16). Expanding exponents of the type \( \exp\{u_j B_j(x)\} \) with the appropriate derivatives of \( B_j(x) \) allows transformation of (15) into a sum of integrals.25 Doing so leads to the following series representation for \( \sigma(\theta, \phi) \):

$$
\sigma(\theta, \phi) = \sigma_{KA} + \frac{(k_1 \cdot v)^2 (kl)}{2 \sqrt{\pi k_1 v^2}} \exp\{-v^2 k_1^2\} \sum_{m=0}^{\infty} \frac{u_j h^{2m}}{m!} \left[-A_1(m) + A_2(m) + \frac{2a_r^2}{\rho (m+1)^{5/2}} D_4 \left(\frac{\partial_{kl}^r}{\sqrt{2(m+1)}}\right)\right] 
$$

\( \times \exp \left[-\frac{(\partial_{kl}^r)^2}{8(m+1)} + \frac{v_2^2 j^2 (a_r^2 - 2b)}{(m+2)^{5/2}} D_4 \left(\frac{\partial_{kl}^r}{\sqrt{2(m+2)}}\right)\right]\),

(A1)

where \( \sigma_{KA} \) designates the series representation of the Kirchhoff approximation scattering cross section, \( D_4 \) is the fourth-order parabolic cylinder function, \( \gamma = \sqrt{2h/l} \) is the rms surface slope, \( \partial_{kl}^r = u_j / k_1 \), and the following notation has been introduced:

$$
A_0 = 1 + 2av_2^2 + a^2 v_2^2 j^2 + 2b^2 \left(\frac{1}{a_r^2} - v_2^2 j^2\right), \quad (A2)
$$

$$
A_1(m) = \frac{1}{\sqrt{m+1}} \exp\left[-\frac{(\partial_{kl}^r)^2}{4(m+1)}\right] \left[A_{11} + A_{12} \left(\frac{(k_1^2)}{2(m+1)} - \frac{\partial_{kl}^r (k_1^2)}{4(m+1)^2}\right)\right], \quad (A3)
$$

$$
A_2(m) = \frac{v_2^2 j^2 (a_r^2 - 2b)}{\sqrt{m+2}} \exp\left[-\frac{(\partial_{kl}^r)^2}{4(m+2)}\right] \left[1 - \frac{1}{2(m+2)} + \frac{(\partial_{kl}^r)^2}{4(m+2)^2}\right], \quad (A4)
$$

$$
A_{11} = -2a_r^2 \left[ v_r - a \left(\frac{3}{a_r^2} - \gamma^2 g_r \left(1 - \frac{2b}{a_r^2}\right)\right)\right], \quad (A5)
$$

$$
A_{12} = \frac{a_r^2}{k_1^2} \left[v_r - a \left(\frac{6}{a_r^2} - \gamma^2 g_r \left(1 - \frac{2b}{a_r^2}\right)\right)\right]. \quad (A6)
$$

All other variables are identified in the text. Equation (A1) is computationally more efficient than the integral form given by (15) when \( u_j h \) is relatively small and \( \partial_{kl}^r k_1 \) is relatively large.


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24E. Thorsos (personal communication). As $k\ell$ is increased, a larger ensemble size is required for the integral equation result to be accurate. See Section III C of Ref. 22.